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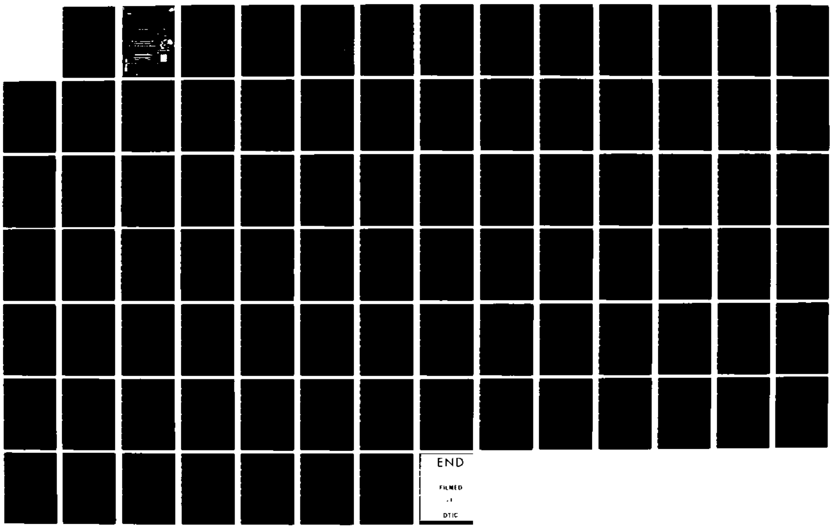

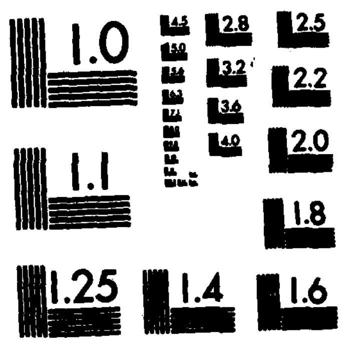

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MAXIMUM LIKELIHOOD ESTIMATION IN MONOPULSE PROBLEMS: PART I: THE STRUCTURE OF THE ESTIMATORS

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ABSTRACT

Maximum likelihood methods are applied to a series of monopulse problems, involving both angle estimation and signal detection. Only the two-beam, off-boresight monopulse problem is studied. Explicit maximum likelihood estimators are obtained in Part I, and their probability distributions will be discussed in the forthcoming Part II. Both deterministic and stochastic signal models are used here, and maximum likelihood estimates are obtained for the single pulse case and for different models of the multiple pulse problem. Particular emphasis is given to the problem of angle estimation in correlated noise, representing the case of arbitrary interference.

CONTENTS

PART I: STRUCTURE OF THE ESTIMATORS

1. INTRODUCTION

In classical monopulse, a receiver has two matched channels, each connected to its own antenna. The antennas have different patterns, or different phase centers, and signals of known waveform arrive from an unknown direction which is to be measured by the receiver system. This entire study is directed at the rather special case in which the signal direction is characterized by a single unknown parameter, such as azimuth or elevation. This restricted problem is interesting because of its long history and also because it lends itself to a fairly complete mathematical analysis in an interesting way.

The results developed here have practical application to direction finding systems in which only one component of direction is unknown, and also to the case where the antenna gains are assumed insensitive to the other component, as with a radar using fan beams. In systems measuring both azimuth and elevation, the techniques discussed here can be used separately on each component, with somewhat suboptimum results.

The several estimation problems analyzed here are modeled and solved in rather general terms. However, the basic monopulse problems which motivated the work are spelled out in enough detail to keep the results in direct touch with practical applications. The method of maximum likelihood is used throughout, and the focus is on fixed-sample angle estimation (i.e., off-bore-sight monopulse), and not on angle tracking, as such.

A mathematical setting has been developed for the special problem of two-beam, one-parameter monopulse, which appears to be unique to this case. Instead of dealing with beam gain ratios and complex data sample ratios as points in a complex plane, this plane is stereographically mapped onto the unit sphere. The likelihood function is directly related to distance on the surface of this sphere, so that maximum likelihood estimates have very simple interpretations as geometrical projections. Transformations of coordinates, of which the well-known transformation of amplitude-comparison to phase-comparison monopulse is an example, correspond to rigid rotations of the sphere, and a slightly more general class of transformations allows for the "whitening" of problems involving external interference of any kind.

The remaining sections of Part I deal first with the simple case of one sample pair and white noise, where the spherical model is introduced and illustrated in terms of standard amplitude-comparison and phase-comparison systems. Transformations are discussed next, and it is shown that the amplitude-comparison and phase-comparison formulations are different versions of the same general problem, viewed in special coordinate systems. Generalizations are then introduced along two lines. In one line, multiple sample-pairs are considered, with coherent and incoherent signal models. In the other line, non-white noise is introduced, permitting the analysis of direction-finding in the presence of interference. Finally, a non-deterministic signal model is used to show that the same estimators for direction of arrival are obtained as in the deterministic models of the previous sections.

In Part II, to be published separately, the performance of the maximum likelihood estimators is discussed. Explicit probability density functions are given there for the single pulse case, for both internal and external sources of noise. For the multiple pulse case, a recursive expression for the probability density function is given, together with explicit formulas for certain moments which can be used to characterize accuracy.

Many of the results in this report have been obtained by, or in collaboration with, J. R. Johnson. His insight and encouragement has played an essential role in the development of the ideas presented here.

2. GENERAL FORMULATION OF THE CLASSICAL TWO-BEAM MONOPULSE PROBLEM

In a two-channel monopulse system we model the complex modulation functions at the respective antenna terminals as $Z_1(t)$ and $Z_2(t)$, where

$$Z_k(t) \equiv Ae^{i\delta}\, S(t)V_k(\gamma) + W_k(t) \quad ,$$

for $k = 1,2$. The rf amplitude and phase are represented by A and δ, while S(t) stands for the complex signal waveform, assumed known. Azimuth (or elevation) is called γ, and the $V_k(\gamma)$ are the complex voltage gains of the two antennas in direction γ. Finally, $W_1(t)$ and $W_2(t)$ are totally independent, complex circular white noise processes, each with zero mean and single-sided power spectral density N_o. This last assumption results in the complex covariance function

$$E\ W_k(t)W_k^*(t') = 2N_o\ \delta(t'-t)$$

for each noise process, and the circular property is represented by the equation

$$E\ W_k(t)W_k(t') = 0$$

for k = 1 or 2.

These signals are passed through identical filters, matched to the expected modulation, so that the complex outputs, sampled at the correct time for maximum signal components, are

$$z_k = G \int S^*(t)Z_k(t)dt \equiv s_k + n_k \quad .$$

The real and imaginary parts of each complex sample represent in-phase and quadrature components, and G is a gain factor. The separate signal and noise terms are, of course,

$$s_k = G\ Ae^{i\delta}\ V_k(\gamma) \int |S(t)|^2\ dt$$

and $$n_k = G \int S^*(t) W_k(t) dt \quad ,$$

and signal parameters have been suppressed in the notation.

The noise samples are complex, circular Gaussian variables, and we have the expression

$$E|n_k|^2 = 2G^2 N_o \int |S(t)|^2 dt$$

for the variances (k=1,2).

We now choose the filter gain factor to make

$$G^2 N_0 \int |S(t)|^2 dt = 1 \quad ,$$

so that

$$E|n_k|^2 = 2 \quad .$$

Thus the real and imaginary parts of n_1 and n_2 constitute four real Gaussian variables, each with zero mean and unit variance. Next, we introduce the normalized voltage gains

$$v_k(\gamma) \equiv \frac{V_k(\gamma)}{\sqrt{|V_1(\gamma)|^2 + |V_2(\gamma)|^2}} \quad ,$$

and write the signal components in the form

$$s_k = b v_k(\gamma) \quad ,$$

where b is our last new parameter, representing complex amplitude, and given by

$$b = G A e^{i\delta} \sqrt{|V_1(\gamma)|^2 + |V_2(\gamma)|^2} \int |S(t)|^2 dt \quad .$$

The argument of b is still the rf phase, and the magnitude is determined from

$$|b|^2 = \frac{A^2}{N_0} \{|v_1(\gamma)|^2 + |v_2(\gamma)|^2\} \int |S(t)|^2 dt ,$$

as a result of our choice of G. But

$$\frac{1}{2} A^2 |v_k(\gamma)|^2 \int |S(t)|^2 dt$$

is the total signal energy in the k^{th} channel at the antenna terminals, and if the sum of these two energies is called E_s, the total signal energy collected by the system, then we have

$$|b|^2 = \frac{2E_s}{N_0} ,$$

and the signal to noise ratio of the system may be defined as

$$\mathrm{SNR} = \frac{|s_1|^2 + |s_2|^2}{E|n_1|^2 + E|n_2|^2} = \frac{|b|^2}{4} = \frac{E_s}{2N_0} .$$

Everything that follows is based on the simple model

$$z_k = bv_k(\gamma) + n_k ,$$

with normalized beam gains and univariate noise components, as defined above. It should be clear that the special assumptions of ideal matched filtering and sampling can be relaxed, with only a change in the significance of b. In Part I we are concerned only with the estimation of γ, and b is a nuisance parameter, but the connection with signal energy and noise spectral density is important in Part II, where performance is evaluated and signal to noise ratio is a fundamental quantity. It is then important to relate b to physical parameters by means of a specific model, as we have just illustrated above.

Note that no assumptions have been made with respect to the voltage gains of the antennnas, and that the basic model could apply to any problem characterized by sample pairs with relative signal components fixed by a single real parameter.

The probability density function for the two complex samples is

$$f(z_1,z_2) = \frac{1}{4\pi^2} \exp\left(-\frac{L^2}{2}\right) \quad ,$$

where

$$L^2 = |z_1 - bv_1(\gamma)|^2 + |z_2 - bv_2(\gamma)|^2 \quad .$$

Maximum likelihood (ML) estimation of b and γ is, of course, equivalent to finding the values which minimize L^2. When we expand this expression, using the normalization property of the v_k, and complete the square we find

$$L^2 = |z_1|^2 + |z_2|^2 + |b - [v_1^*(\gamma)z_1 + v_2^*(\gamma)z_2]|^2$$

$$- |v_1^*(\gamma)z_1 + v_2^*(\gamma)z_2|^2 \quad .$$

The estimate of b, given γ, is of course

$$\hat{b}(\gamma) \equiv v_1^*(\gamma)z_1 + v_2^*(\gamma)z_2 \quad ,$$

and $\hat{\gamma}$ must minimize

$$\underset{b}{\text{Min}}\, L^2 = |z_1|^2 + |z_2|^2 - |v_1^*(\gamma)z_1 + v_2^*(\gamma)z_2|^2 \quad ,$$

after which $\hat{b} = \hat{b}\,(\hat{\gamma})$.

Again we expand, and write

$$\min_b L^2 = (|z_1|^2 + |z_2|^2)\{1- \frac{|v_1(\gamma)|^2|z_1|^2+|v_2(\gamma)|^2|z_2|^2+ 2\mathrm{Re}[v_1^*(\gamma)v_2(\gamma)z_1z_2^*]}{|z_1|^2 + |z_2|^2}\}$$

The quantity in curly brackets will determine the estimator of γ, and it is clear that this expression depends on the data samples only through the complex ratio z_1/z_2 . Recalling the normalization, it is also obvious that this quantity depends on the gain factors through the ratio v_1/v_2 only. We express these ratios in terms of four, real angle variables, as follows:

$$\frac{v_1(\gamma)}{v_2(\gamma)} \equiv \tan(\theta/2)e^{i\phi}$$

$$\frac{z_1}{z_2} \equiv \tan(\eta/2)e^{i\psi} \quad .$$

The angles θ and η are restricted to the range $[0,\pi]$, so that ϕ and ψ are the arguments of the respective ratios. Angles θ and ϕ depend implicitly on γ, of course, and we may interpret θ and η as polar angles, ϕ and ψ as azimuthal angles, describing points on the unit sphere. The data ratio is mapped into a single point (η,ψ), but the gain ratio traces out some trajectory on the sphere, as γ varies over its normal range.

The direction finding properties of the actual system are completely specified by the "characteristic trajectory" over which (θ,ϕ) varies, and there is no guarantee, in general, that this trajectory is a simple curve. It is also not guaranteed that any point on the trajectory corresponds to a single value of γ, but it is clear from the form of L^2 that there will be no way to resolve these ambiguities from the data samples, z_1 and z_2.

It is easy to show that

$$|v_1(\gamma)|^2 = \sin^2(\theta/2) = \frac{1}{2}(1 - \cos\theta) ,$$

$$|v_2(\gamma)|^2 = \cos^2(\theta/2) = \frac{1}{2}(1 + \cos\theta), \quad \text{and that}$$

$$v_1(\gamma)v_2^*(\gamma) = \sin(\theta/2)\cos(\theta/2)e^{i\phi} = \frac{1}{2}\sin\theta \, e^{i\phi} .$$

Similarly,

$$\frac{|z_1|^2}{|z_1|^2 + |z_2|^2} = \frac{1}{2}(1 - \cos\eta) ,$$

$$\frac{|z_2|^2}{|z_1|^2 + |z_2|^2} = \frac{1}{2}(1 + \cos\eta) , \quad \text{and}$$

$$\frac{z_1 z_2^*}{|z_1|^2 + |z_2|^2} = \frac{1}{2}\sin\eta \, e^{i\psi} .$$

When these expressions are substituted we obtain

$$\operatorname*{Min}_{b} L^2 = \frac{1}{2}(|z_1|^2+|z_2|^2)[1 - \cos\theta \cos\eta - \sin\theta \sin\eta \cos(\phi - \psi)].$$

But

$$\cos\theta \cos\eta + \sin\theta \sin\eta \cos(\phi - \psi) = \cos\Delta ,$$

where Δ is the arc distance between (θ,ϕ) and (η,ψ) on the surface of the sphere. Finally,

$$\operatorname*{Min}_{b} L^2 = (|z_1|^2 + |z_2|^2) \sin^2(\Delta/2) \quad ,$$

and the estimation problem reduces to minimizing Δ, or finding that point on the characteristic trajectory of the system which is closest (in the ordinary metric sense on the sphere) to the data point (η,ψ) .

If the characteristic trajectory is complicated, then the spherical interpretation is of no particular help in finding an algorithm for explicit minimization, but in some important idealizations, it permits a simple, explicit answer to the estimation problem.

The equation

$$w = \tan(\theta/2)e^{i\phi}$$

establishes a mapping of the complex w-plane onto the unit sphere and vice versa. It is, in fact, the well-known stereographic projection of the sphere onto the plane. The plane may be taken as the extension of the sphere's equatorial plane, with points geometrically projected between sphere and plane along lines originating at the south pole. The mapping is conformal and also maps circles into circles. We do not need any of these facts (although they help the intuition), since it is more fundamental to this problem to concentrate on the many-to-one mapping established between complex two-component vectors, such as

$$\begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} ,$$

and the corresponding points (η,ψ) on the sphere.

In addition to the estimation of signal parameters, it is often necessary to decide on the presence or absence of any signal in the first place. The "noise-alone" hypothesis, $b=0$, is characterized by the sample probability density function

$$f_0(z_1,z_2) = \frac{1}{4\pi^2} \exp\left(-\frac{L_0^2}{2}\right) ,$$

where

$$L_0^2 = |z_1|^2 + |z_2|^2 .$$

Maximum-likelihood testing then reduces to evaluation of the test statistic $L0^2 - \underset{b,\gamma}{\mathrm{Min}}\, L^2$. If the ML estimate, $\hat{\gamma}$, corresponds to the minimum distance $\hat{\Delta}$, then

$$\underset{b,\gamma}{\mathrm{Min}} \; L^2 = \frac{1}{2} (|z_1|^2 + |z_1|^2)(1 - \cos\hat{\Delta}),$$

and detection is based on the test

$$\frac{1}{2} (|z_1|^2 + |z_2|^2)(1 + \cos \hat{\Delta}) > \text{threshold} .$$

Examples of the ML estimation of γ and the corresponding detection criteria are given in the next two sections, beginning with the familiar examples of phase-comparison and amplitude-comparison monopulse.

Before leaving the topic of detection, it is interesting to consider a refinement sometimes called interference detection, or "data editing". Monopulse measurements are easily disturbed by extraneous signals, since the estimation procedure depends critically on the assumption of a single source, with the consequent cancellation of the unknown signal amplitude and phase in the ratio, z_1/z_2 , upon which estimation is based. The presence of another signal component can be detected, however, and this fact used as a flag to inhibit processing of the affected data for signal location.

Interference detection is accomplished by adding a third hypothesis, "multiple targets", to the single-target and noise-only hypotheses already discussed. If two signals are present, the pdf of the sample pair has the standard form

$$f(z_1,\ z_2) = \frac{1}{4\pi^2}\ e^{-(L_2/2)} \quad ,$$

which applies to all the hypotheses, but in this case L^2 will be

$$L^2 = |z_1 - bv_1 - b'v_1'|^2 + |z_2 - bv_2 - b'v_2'|^2 \ .$$

Here b and b' are complex amplitudes, v1 and v2 are voltage gains for the first signal component, while v1' and v2' refer to the other. The second term can be made to vanish by the choice

$$b' = \frac{1}{v_2'}(z_2 - bv_2) \quad ,$$

leaving

$$\left|z_1 - \frac{v_1'}{v_2'} z_2 + bv_2 \left(\frac{v_1'}{v_2'} \quad \frac{v_1}{v_2}\right)\right|^2$$

for the remaining term. By choosing signal locations at different points on the characteristic, so that

$$\frac{v_1'}{v_2'} \neq \frac{v_1}{v_2} \quad ,$$

we can make this term vanish by a suitable choice for b. Further signal components are superfluous, hence the multiple target hypothesis always allows us to make Min $L^2 = 0$.

For any hypothesis concerning the signal structure, we can define the log likelihood function

$$\Lambda \equiv \log \mathrm{Max}\ f(z_1,\ z_2) = -\frac{1}{2}\,\mathrm{Min}\ L^2 - \log\,(4\pi^2) \ ,$$

where Max and Min refer to signal parameters contained in L^2. For the noise-alone hypothesis, H0, there are no signal components, hence no parameters, and we have

$$\Lambda_0 = -\frac{1}{2} L_0^2 - \log(4\pi^2) \quad ,$$

where, as before,

$$L_0^2 = |z_1|^2 + |z_2|^2 \quad .$$

When one signal component is allowed (hypothesis H_1), confined to the characteristic trajectory of the system, we have found that

$$\text{Min } L^2 = (|z_1|^2 + |z_2|^2) \sin^2(\hat{\Delta}/2) \quad ,$$

where $\hat{\Delta}$ is the minimum distance attained by the ML estimate of location, and hence we can write

$$\Lambda_1 = -\frac{1}{2} L_0^2 \sin^2(\hat{\Delta}/2) - \log(4\pi^2) \ .$$

Finally, for the multiple-target hypothesis, H2, we have

$$\Lambda_2 = - \log (4\pi^2),$$

since we have shown that Min $L^2 = 0$ in this case.

In pairwise hypothesis testing, such as H_1 verus H_0, we compare the difference of the log likelihood functions to a threshold. Thus, we accept H1 over H0 if

$$\Lambda_1 - \Lambda_0 > \mu \quad ,$$

for some constraint μ, which will be chosen to meet the false alarm probability requirements. The resulting test, in this case, has already been discussed. To generalize the testing to multiple hypotheses, we introduce a constant, μ_k, for each hypothesis, and consider the numbers $\Lambda_k - \mu_k$, for k=0, 1 and 2. If the largest of these occur for $k = \ell$, then hypothesis H_ℓ is chosen. This procedure generalizes pairwise testing in a natural way, and (with an arbitrary tie-breaking rule) leads to an unambiguous choice of hypothesis in every instance. This procedure can also be viewed as a special case of Bayes hypothesis testing with a suitable cost matrix.

Since only differences of the μ_k affect the decision process, we can take the largest of the numbers

$$T_0 = -\frac{1}{2} L_0^2 - \mu_0 \quad ,$$

$$T_1 = -\frac{1}{2} L_0^2 \sin^2(\hat{\Delta}/2) - \mu_1 \;, \quad \text{and}$$

$$T_2 = -\mu_2 \quad ,$$

as the basis for decision. Finally, we can add μ_2 to each of these quantities and base decision on the smallest of the numbers

$$t_0 \equiv L_0^2 - C_0 \quad ,$$

$$t_1 \equiv L_0^2 \sin^2(\hat{\Delta}/2) - C_1 \;, \text{ and}$$

$$t_2 \equiv 0.$$

We have introduced new threshold constants, $C0 = 2\ (\mu 2 - \mu 0)$ and $C1 = 2(\mu 2 - \mu 1)$.

It is interesting that L_0^2 and $\hat{\Delta}$ are sufficient statistics for the three-way decision process. L_0^2 is a measure of the total energy contained in the two samples, and $\hat{\Delta}$ is a measure of how closely the data ratio, z_1/z_2 , resembles an ideal signal ratio, i.e., a point on the characteristic trajectory. With two signals present, the data ratio can be anything, even in the absence of noise. If this ratio falls on the characteristic by accident, the data will look like a single target, and there is no reason to expect the receiver to decide otherwise. But usually, the data ratio will off the characteristic when an interferor is present, and $\hat{\Delta}$ will allow us to sense this, even though the received energy is large. In a simple test of H1 versus H0, the receiver will choose H_1 if the received energy is large enough, even the estimated location is far from the system trajectory.

To see how the decision mechanism operates, we can define the equivalent pair of statistics

$$X \equiv L_0 \cos(\hat{\Delta}/2) ,$$

$$Y \equiv L_0 \sin(\hat{\Delta}/2) , \text{ and}$$

plot the decision regions in the X-Y plane. Since distances between points on a sphere never exceed π, both X and Y are non-negative. The H0:H1 boundary is given by t0 = t1 , or

$$L_0{}^2 \cos^2(\hat{\Delta}/2) = C_0 - C_1 .$$

Obviously, C0 must exceed C1 or we could never have t0 less than t1. Hence the required boundary is the line

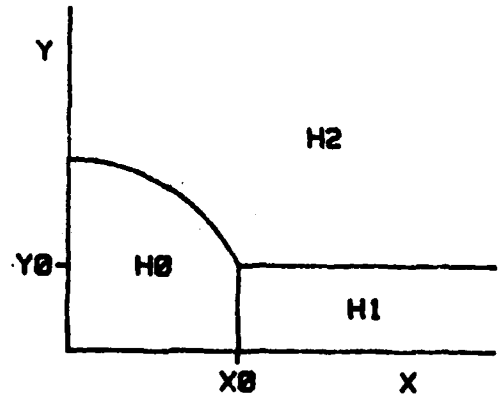
$$X = \sqrt{C_0 - C_1} \equiv X_0 .$$

Similarly, the H1:H2 boundary is the line

$$Y = \sqrt{C_1} \equiv Y_0 ,$$

and C1 is positive to allow the possibility of having $t1 < t2$. Finally, the H0:H2 boundary is the circular arc $\sqrt{X^2 + Y^2} = \sqrt{C0} = \sqrt{X0^2 + Y0^2}$.

The three boundaries meet at the point (X0, Y0), and a typical situation would be as shown in the figure, where each region is labeled by the hypothesis chosen for corresponding data points. Only those segments of the boundaries where the first choice changes are shown.



In general, small sample energy results in the "noise-alone" decision, while large energy implies either H_1 or H_2. For H_1 to be preferred, the angle $\hat{\Delta}$ must be small (the data point must be near the system trajectory), and the larger the received energy, the smaller $\hat{\Delta}$ must be. The parameters X0 and Y0 allow some control over the performance of the system, as characterized by errors such as false alarms and incorrect editing.

In Section 7 we deal with the problem of interference in another way. The interfering sources are treated as stationary noise sources, and it is assumed that the resulting noise covariance matrix is known. Maximum likelihood estimation is then performed in the context of known, non-white noises. That approach makes sense when one has an opportunity to observe the noise and then attempt the estimation of target location before the noise has changed statistically. The treatment in the present section is complimentary to this, dealing with intermittent and unpredictable interference in a simple way, by recognizing and discarding contaminated samples.

3. IDEAL PHASE-COMPARISON AND IDEAL AMPLITUDE-COMPARISON MONOPULSE

A phase-comparison monopulse system is essentially an interferometer, which employs two identical antennas with separated phase centers. If the complex voltage gain of either antenna, relative to its own phase center, is V, then the corresponding gains relative to a phase center midway between them can be written

$$V1(\gamma) = V e^{ik\cos\gamma}$$

$$V2(\gamma) = V e^{-ik\cos\gamma}$$

The gain function V depends implicitly on the actual direction of arrival of the signal, and it is assumed that the antenna boresight directions are parallel. Our parameter γ is the angle between the signal direction of arrival and the baseline direction established by the line between the separate antenna phase centers. The constant k, of course, is $\pi\, d/\lambda$, where d is antenna spacing and λ is the signal carrier wavelength.

The normalized gain functions are

$$v_1(\gamma) = \frac{1}{\sqrt{2}} e^{ik\cos\gamma + \alpha}$$

$$v_2(\gamma) = \frac{1}{\sqrt{2}} e^{-ik\cos\gamma + \alpha}$$

where α is the argument of V. Only γ can be measured by the system, of course, and the gain ratio depends on this directional parameter alone:

$$\frac{v_1(\gamma)}{v2(\gamma)} = e^{2ik\cos\gamma} \quad .$$

The characteristic trajectory of this system is, of course, the equator (or part of it) on the unit sphere:

$$\theta = \pi/2$$

$$\phi = 2\, k \cos\gamma$$

If d exceeds γ, the whole equator is possible and γ is a multivalued function of ϕ. As mentioned in Section 2, this kind of ambiguity cannot be resolved by the system, and the best we can do is to estimate ϕ. By restricting our discussion to the estimation of ϕ, we are dealing with all phase-comparison sytems, and the final translation of this estimator and its performnce back to the parameter γ will not be carried out. We also choose not to discuss the case $d < \gamma$ explicitly, and define the "ideal phase-comparison system" as one whose characteristic trajectory coincides with the equator of the unit sphere.

For the ideal phase-comparison system, we obtain immediately the well-known result that the ML estimator of ϕ is

$$\hat{\phi} = \psi \doteq \arg(z_1/z_2) = \arg(z_2{}^*z_1) \quad ,$$

since the projection of a point on the sphere to its equator is along a meridian, preserving azimuthal angle, or longitude. Since $\hat{\Delta} = |\eta - \frac{\pi}{2}|$ in this case, we also obtain the detection criterion

$$\frac{1}{2}(|z_1|^2 + |z_2|^2)(1 + \sin\eta) > \text{const.}$$

But

$$(|z_1|^2 + |z_2|^2)\sin\eta = 2|z_1z_2{}^*| \quad ,$$

and therefore detection is based upon the equivalent rule

$$|z_1| + |z_2| > \text{const.}$$

In amplitude-comparison monopulse, two antennas with different patterns, but sharing a common phase center, are employed. In standard notation, the gain of the "sum beam" is Σ, and that of the "difference beam" is Δ. We take $V_1 = \Delta$ and $V_2 = \Sigma$, so that

$$\frac{v_1(\gamma)}{v_2(\gamma)} = \frac{v_1}{v_2} = \frac{\Delta}{\Sigma} \quad .$$

The angle γ is usually azimuth, and the key assumption is that the gain ratio is a real function of γ. This is generally true in practical amplitude-comparison systems within the sum beam main lobe, and we define the "ideal amplitude-comparion system" to be one for which the gain ratio assumes all real values as γ varies. Ambiguities are again possible, but the only measurement supported by the system is an estimation of the value of this real ratio. Usually, Δ is an odd function of γ and Σ is even, hence we assume that $\gamma = 0$ corresponds to a zero value of the ratio.

The characteristic trajectory of an ideal amplitude-comparison system is obviously a great circle through the poles of the unit sphere, composed of the meridians $\phi = 0$ and $\phi = \pi$. It is more convenient to use the equivalent description $\phi = 0$, allowing θ to range from $-\pi$ to $+\pi$, and then

$$\frac{v_1(\gamma)}{v_2(\gamma)} = \tan(\theta/2) \quad .$$

As before, we study amplitude-comparison systems in general by dealing only with the estimation of θ. The properties of individual antenna systems enter only when θ is expressed in terms of γ. This relation is usually simple, and near boresight is often well-approximated by a linear dependence of θ on γ.

It is instructive to rederive the known results for amplitude-comparison monopulse from our formulation. The estimate $\hat{\theta}$ is, of course, obtained by dropping a perpendicular from the data point (η, ψ) to the characteristic meridian. By elementary spherical trigonometry, the result is determined from the equation

$$\tan\hat{\theta} = \tan\eta\cos\psi$$

It is convenient to restrict ψ to the hemisphere

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2} ,$$

and let η range over $[-\pi, \pi]$, in which case it will be found that $\hat{\theta}$ is uniquely determined by assigning it to the same quadrant as η. It is also seen that the distance, $\hat{\Delta}$, to the characteristic never exceeds $\frac{\pi}{2}$ and is determined (using the Law of Sines) by

$$\sin \hat{\Delta} = |\sin \eta \sin \psi|.$$

Amplitude-comparison monopulse is usually discussed in terms of the gain ratio, as a real parameter u, and the data ratio, as a complex variable w. Then

$$\hat{u} = \tan (\hat{\theta}/2)$$

will be expressed in terms of

$$w = \tan (\eta/2)e^{i\psi} .$$

This connection can be derived by noting that

$$\tan \hat{\theta} = \frac{2 \hat{u}}{1 - (\hat{u})^2}$$

and that

$$\frac{2 \operatorname{Re}(w)}{1 - |w|^2} = \frac{2 \tan (\eta/2)\cos \psi}{1 - \tan^2(\eta/2)} = \tan \eta \cos \psi .$$

The solutions of the resulting equation:

$$\frac{\hat{u}}{1 - (\hat{u})^2} = \frac{\operatorname{Re}(w)}{1 - |w|^2} ,$$

are

$$\hat{u}_{\pm} = \frac{|w|^2 - 1 \pm \sqrt{(|w|^2 - 1)^2 + 4\,\mathrm{Re}^2(w)}}{2\,\mathrm{Re}\,(w)} \quad .$$

But

$$\mathrm{Re}^2(w) = \frac{1}{2}\ |w|^2 + \frac{1}{2}\ \mathrm{Re}\,(w^2) \quad ,$$

and hence $(|w|^2-1)^2 + 4\,\mathrm{Re}^2(w) = |w|^4 + 1 + 2\,\mathrm{Re}\,(w^2) = |w^2+1|^2$,

which gives

$$\hat{u}_{\pm} = \frac{|w|^2 - 1 \pm |w^2 + 1|}{2\,\mathrm{Re}(w)}$$

The sign ambiguity is resolved by considering the case

$$w = x = \text{real.}$$

Then $\hat{u}_{+} = x$ and $\hat{u}_{-} = -1/x$, and only the positive root is consistent with the constraint of keeping $\hat{\theta}$ and η in the same quadrant. Thus

$$\hat{u} = \frac{|w^2 + 1| + |w|^2 - 1}{2\,\mathrm{Re}\,(w)} = \frac{2\,\mathrm{Re}\,(w)}{|w^2 + 1| + 1 - |w|^2} \quad ,$$

the desired result [1].

As for the detection criterion, we note that

$$\sin\eta\ e^{i\psi} = \frac{2\ z_1 z_2^*}{|z_1|^2 + |z_2|^2} = \frac{2\ w}{1 + |w|^2} \quad ,$$

so that

$$\sin\hat{\Delta} = |\sin\eta\ \sin\psi| = \left|\frac{2\ \mathrm{Im}\,(w)}{1 + |w^2|}\right| \ .$$

Then

$$\cos\hat{\Delta} = \frac{(1+|w|^2)^2 - 4\,\mathrm{Im}^2(w)}{1+|w^2|} = \frac{|1+w^2|}{1+|w|^2} ,$$

as a consequence of the identity

$$\mathrm{Im}^2(w) = \frac{1}{2}|w|^2 - \frac{1}{2}\mathrm{Re}(w^2) ,$$

and we obtain again the known result(2) for the detection statistic:

$$\frac{1}{2}(|z_1|^2 + |z_2|^2)(1+\cos\hat{\Delta})$$

$$= \frac{1}{2}(|z_1|^2 + |z_2|^2)\left\{1 + \frac{\left|1+\frac{z_1^{\,2}}{z_2^{\,2}}\right|}{1+\left|\frac{z1}{z_2}\right|^2}\right\}$$

$$= \frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_1^{\,2} + z_2^2|) .$$

The approximation $\hat{u} \approx \mathrm{Re}\,(w)$ is often used in applications of amplitude-comparison monopulse. The approximation is good whenever $|w|<<1$ (often the case for a signal near boresight) and it is exact, as we have seen, when w happens to be real. In terms of the sphere, the approximation takes the form

$$\tan(\hat{\theta}/2) \approx \tan(\eta/2)\cos\psi ,$$

which is not a natural one to make in this context.

The analysis given in this section is aimed at developing familiarity with the formulation of monopulse problems on the unit sphere. The idealizations made here, of two special great-circle characteristic trajectories, are commonly made, in some form, in the conventional analyses of these problems. They are often acceptable because they fail significantly only outside the range in γ at which the system normally operates. One usually does not expect amplitude-comparison monopulse to work in the antenna backlobes, for example, and past tracking data is almost always employed in some way to restrict operation to a well-behaved, main-lobe portion of the coverage which also permits unambiguous conversion of γ to actual signal direction.

4. TRANSFORMATION OF COORDINATES IN MONOPULSE ANALYSIS

It has been known for a long time that the analysis of an amplitude-comparison system with sum-beam output Σ and difference-beam output Δ can be transformed into the analysis of a phase-comparison system by the introduction of the new quantities

$$S_{\pm} = \frac{1}{\sqrt{2}} (\Sigma \pm i\Delta).$$

The signal components of the new "coordinates" are equal in magnitude and the noise components are again independent with the same variances as before. Formulated in terms of S_+ and S_-, the problem is mathematically identical to one of pure phase-comparison monopulse. This is not, of course, an accident. In this Section we introduce a class of simple, linear transformations of the data pair, z_1 and z_2, and investigate the induced transformation of coordinates on the unit sphere. It turns out that data transformations which preserve the "white" character of the noise samples (independence and equivariance) induce rigid rotations of the sphere and, in particular, the transformation from (Σ,Δ) to (S_+, S_-) can be derived as the one necessary to rotate the characteristic trajectory of ideal amplitude-comparison monopulse into the equatorial characteristic of its pure-phase counterpart.

We consider the samples, z_1 and z_2, to be components of a vector, z, in a two-dimensional complex space, writing

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} .$$

Similarly, the normalized gains and the noise samples are components of vectors v and n, and we have

$$z = bv + n .$$

The conjugate row-vector to z is $z^\dagger$, given by

$$z^\dagger = \begin{bmatrix} z_1^* & z_2^* \end{bmatrix} ,$$

and the covariance matrix of the noise vector is denoted

$$M = Enn^{\dagger}.$$

We have modeled this noise as white, each complex component having variance 2, so that

$$M = 2\ I_2\ ,$$

where I_2 denotes the 2 x 2 unit matrix.

Suppose we define new variables, z_1' and z_2' , as linear transforms of z_1 and z_2 by writing

$$z' = \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = T\ z\ ,$$

where T is a non-singular 2 x 2 matrix with complex components. In terms of z', the system has gain vector $v' = T\ v$, and a noise covariance matrix

$$M' = TMT^{\dagger} = 2TT^{\dagger}\ ,$$

where $T^{\dagger}$ is the complex transpose of T. Obviously, if we choose T to be unitary, i.e., $TT^{\dagger} = I_2$, than the new noise covariance matrix is the same as the old, and moreover the new gain vector is automatically normalized.

To any vector, such as z, there corresponds a point (η,ψ) on the unit sphere, determined by the equation

$$\tan(\eta/2)e^{i\phi} = \frac{z_1}{z_2}\ .$$

This correspondence is many-to-one, since z can be multiplied by a complex scalar without changing η and ψ. The transformed vector, $z' = Tz$, determines another point (η', ψ') on the sphere, through

$$\tan(\eta'/2)e^{i\psi'} = \frac{z_1'}{z_2'} \quad ,$$

and hence the transformation T induces a mapping of the sphere onto itself. This mapping is given explicitly by the equation

$$\tan(\eta'/2)e^{i\psi'} = \frac{T_{11}\tan(\eta/2)e^{i\psi} + T_{12}}{T_{21}\tan(\eta/2)e^{i\psi} + T_{22}} \quad ,$$

in terms of the components of T. This relation will be used repeatedly in this study.

Now it is an established fact that when T is unitary, the induced transformation on the sphere is a simple rotation. This equivalence is easily proved, as follows. Let

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

by a vector in our sample space, and consider the inner product

$$(a,z) \equiv a_1^* z_1 + a_2^* z_2 \quad .$$

If

$$\frac{a_1}{a_2} \equiv \tan(\theta/2)e^{i\phi} \quad ,$$

Then

$$\frac{|a_1|^2}{|a_1|^2 + |a_2|^2} = \sin^2(\theta/2) \ ,$$

$$\frac{|a_2|^2}{|a_1|^2 + |a_2|^2} = \cos^2(\theta/2) \ , \quad \text{and}$$

$$\frac{2a_1a_2^*}{|a_1|^2 + |a_2|^2} = \sin\theta e^{i\phi} \ ,$$

exactly like the relations used in Section 2. Then, writing $||a||^2 = (a,a)$ and $||z||^2 = (z,z)$ for the norms of these vectors, we compute

$$|(a,z)|^2 = |a_1|^2|z_1|^2 + |a_2|^2 |z_2|^2 + 2\,\mathrm{Re}\{a_1^* a_2 z_1 z_2^*\}$$

$$= ||a||^2||z||^2 \left[\frac{1-\cos\theta}{2}\ \frac{1-\cos\eta}{2} + \frac{1+\cos\theta}{2}\ \frac{1+\cos\eta}{2} + \frac{1}{2}\sin\theta\sin\eta\ \mathrm{Re}\{e^{i(\psi-\phi)}\}\right] .$$

In terms of Δ , the distance on the surface of the sphere between the points (θ,ϕ) and (η,ψ) , we have

$$|(a,z)|^2 = ||a||^2||z||^2\ \frac{1+\cos\Delta}{2} \ ,$$

since

$$\cos\Delta = \cos\theta\cos\eta + \sin\theta\sin\eta\cos(\phi - \psi) \ .$$

Now suppose we introduce a unitary transformation, V, and define the new vectors $a' = Va$ and $z' = Vz$. To these vectors correspond points $(\theta', \phi',)$ and (η', ψ') on the sphere, and obviously

$$|(a',z')|^2 = ||a'||^2||z'||^2 \quad \frac{1 + \cos\Delta'}{2} \quad ,$$

where Δ' is the distance between (θ', ϕ') and (η', ψ'). But a unitary transformation leaves inner products and norms unchanged, and we can immediately conclude that $\Delta' = \Delta$, so the transformation of the sphere has preserved the distance between the original two points. Since these points were arbitrary, the transformation must be a rotation.

The fact of this equivalence is all that is needed in this entire analysis, although it is useful to know exactly which unitary matrix corresponds to a prescribed rotation. A simple derivation of this detailed correspondence appears in Appendix A, where the amplitude-comparison /phase-comparison transformation is given as an example.

From an analytical point of view, ideal phase-comparison and amplitude-comparison systems are versions of the same basic problem, expressed in different coordinate systems. Since any rotation of the sphere can be effected by a simple unitary change of basis in the sample data space, we can say that all monopulse systems are equivalent whose characteristic trajectories are great circles on the sphere. We refer to these as "great circle systems", and say we are working in "ideal phase-comparison coordinates" when we have rotated the characteristic trajectory onto the equator. The important portion of a non-ideal trajectory of a real system can be approximated by a great circle arc, and then rotated onto a portion of the equator. The ideal phase-comparison estimator, $\hat{\phi} = \psi$, can then be used as an approximation, and expressed back in terms of the original sample variables of the system.

A broader class, whose importance will appear later, includes all systems whose characteristic trajectories are small circles on the sphere. By rotation, these are all equivalent to systems whose characteristics are circles of constant polar angle (parallels of latitude), say $\theta = \theta o$. It is obvious that the ML estimator of ϕ for such a system is still $\hat{\phi} = \psi$, and that the minimum distance attained by this estimate is $\hat{\Delta} = |\eta - \theta o|$. From this last it is easily shown that the detection criterion is

$$|z_1| \sin(\theta o/2) + |z_2| \cos(\theta_o/2) > \text{threshold} .$$

In Part II it is shown that the accuracy of the ML estimator of θ for this system degrades as the characteristic trajectory recedes from the equator. For this and other reasons it appears that the great circle trajectory is the best that can be attained with a two beam monopulse system.

So far we have emphasized unitary transformations of the data vector because they leave the noise statistically invariant and preserve the normalizations of the gain vector, while moving the characteristic trajectory (whatever it may be) around on the sphere in a simple and useful way. Non-unitary transformations are also important, and they are required in the whitening operation used in the analysis of monopulse in interference, in Section 7. These transformations map the sphere onto itself in a more complicated way, but the mappings are still conformal and circles on the sphere are still transformed into circles, although without the preservation of radius. A great circle trajectory is transformed by a non-unitary, noise-whitening transformation into a small circle, and this explains the appearance of "small circle systems" in the study. The basic theory of non-unitary transformations, as applied to our unit sphere, is the theory of linear fractional transformations of the complex plane. If we write $(z_1/z_2) = w$ and $(z_1'/z_2') = w'$, then clearly $z' = Tz$ implies

$$w' = \frac{T_{11}\, w + T_{12}}{T_{21}\, w + T_{22}} ,$$

as we have already seen. The properties of these transformations are well-known[3], and carry over onto the sphere by means of the conformality and circle-preserving properties of the stereographic projection. Some of the analysis in this study, particularly in Section 7, is motivated by the known properties of linear fractional transformations. The exposition here is self-contained, however, with proofs of statements given in terms of spherical coordinates, when needed.

5. EXTENSIONS TO MULTIPLE-PULSE ANGLE ESTIMATION

In this Section we make two extensions of the analysis, each characterized by the use of multiple sample-pairs as inputs to the estimation problem. Our first example is rather simple, and represents a class of problems exhibiting signal coherence over the set of samples. The point of the example is to illustrate the problem of the simultaneous estimation of angle and other signal parameters, such as Doppler frequency. Without additional parameters, nothing new would be introduced by multiple coherent samples, which is a situation already included in the formulation given, since the original signal waveform was quite general.

We formulate the problem directly in terms of the sequence of sample pairs, $z_1(n)$ and $z_2(n)$, where the subscripts refer to beams, or channels, as before, and n is a sample index running from 1 to N. We postulate that the signal components of these samples have the form

$$Ez_k(n) = b\, S(n) v_k(\gamma) \quad ; \qquad (k = 1,2) \ ,$$

where b is a complex amplitude parameter and $v_1(\gamma)$ and $v_2(\gamma)$ are normalized beam gains, just as before. The new feature is a complex signal sequence, S(n), presumed to depend upon one or more implicit parameters, α, and normalized as follows:

$$\sum_{n=1}^{N} |S(n)|^2 = 1.$$

The motivating example is Doppler modulation, in which a sequence of returned radar pulses is passed through a matched filter, as described in Section 2, with the result that the complex output amplitudes are modulated by a Doppler sequence, such as

$$S(n) = \frac{1}{\sqrt{N}} e^{i\, n\, \alpha} \quad .$$

In all cases the noise is assumed to be white, and independent and statistically identical from sample to sample.

Maximum likelihood estimation of all the unknown parameters immediately reduces to the minimization of

$$L^2 = \sum_{n=1}^{n} \left\{ |z_1(n) - b\ S(n)v_1(\gamma)|^2 + |z_2(n) - b\ S(n)v_2(\gamma)|^2 \right\}.$$

$$= \sum_{n=1}^{n} \left\{ |z_1(n)|^2 + |z_2(n)|^2 + |b|^2 \right\}$$

$$- 2\ \mathrm{Re}\ b^* \left\{ \sum_{n=1}^{n} [v_1^*(\gamma)S^*(n)z_1(n) + v_2^*(\gamma)S^*(n)z_2(n)] \right\} \quad .$$

We have made use of our normalization conventions, and we now introduce the definitions

$$||z_k||^2 \equiv \sum_{n=1}^{n} |z_k(n)|^2 \ ,$$

$$L_o^2 \equiv ||z_1||^2 + ||z_2||^2 \quad ,$$

and

$$f_k \equiv (S, z_k) \equiv \sum_{n=1}^{n} S^*(n) z_k(n) \quad .$$

The unknown parameters represented by α are now implicit in f_1 and f_2. Substituting, we obtain

$$L^2 = L_o^2 + |b|^2 - 2\ \mathrm{Re}\{b^*[v_1^*(\gamma)f_1 + v_2^*(\gamma)\ f_2]\} \quad ,$$

which yields

$$b_1 = v_1^* f_1 + v_2^* f_2$$

and

$$\operatorname*{Min}_{b} L^2 = L_o^2 - |v_1^* f_1 + v_2^* f_2|^2 \quad .$$

The ultimate estimates of α and γ are to be substituted into b_1 to yield $\hat{b}$.

Note that our expression for $\operatorname*{Min}_{b} L^2$ is exactly the same as the one derived in Section 2, except that f_1 and f_2 replace z_1 and z_2. Thus we could interpret the $z_k(n)$ as pairs of time samples of the incoming signals at the antenna terminals. The independent noise samples would then be direct samples of wide band white noise. The f_k are, of course, correlations of the samples with an expected waveform, with or without parameters like Doppler. Then, by a somewhat heuristic limiting process, one could conclude that matched filtering was a good thing to do before proceding with angle estimation. But this was the processing postulated in Section 2, and the point of the present discussion is that matched filtering is a proper part of the overall ML estimation problem, starting with the incident time waveforms.

The characteristic trajectory of the system is the same as before, introduced by way of the definition

$$\frac{v_1(\gamma)}{v_2(\gamma)} = \tan(\theta/2)\, e^{i\phi} \; .$$

But now the data is represented by the point (η, ψ), determined by the definition:

$$\frac{f_1(\alpha)}{f_2(\alpha)} = \tan(\eta/2) e^{i\psi} \quad ,$$

which depends upon α, yet to be estimated. If Δ the arc distance between the points (η,ψ) and (θ,ϕ) are the unit sphere, then we have

$$\operatorname*{Min}_{b} L^2 = L_o^2 - \frac{1}{2}\left(|f_1|^2 + |f_2|^2\right)(1 + \cos\Delta) \quad ,$$

since the expressions are the same as in Section 2 with the f_k replacing the z_k .

With an arbitrary system trajectory, the estimates of γ and α will be coupled in a complicated way, since $\hat{\Delta}$ will be the minimum distance between the characteristic trajectory and the curve or region represented by the points (η, ψ). However, for a simple trajectory, the estimates can be separated and performed in a direct sequential way. Using the small-circle system as an example, the characteristic trajectory is fixed by $\theta = \theta_o$ and for given α, ϕ is estimated as

$$\hat{\phi} = \psi = \arg (f_2^* f_1) \quad ,$$

and $\hat{\Delta} = |\eta - \theta_o|$. Finally, α is estimated by maximizing the expression

$$\frac{1}{2} (|f_1|^2 + |f_2|^2)(1 + \cos \hat{\Delta})$$

$$= \frac{1}{2} (|f_1|^2 + |f_2|^2) + \frac{1}{2} \cos \theta_o (|f_2|^2 - |f_1|^2) + \sin\theta_o |f_1 f_2^*|$$

$$= [\sin (\theta_o/2)|f_1(\alpha)| + \cos(\theta_o/2)|f_2(\alpha)|]^2 \quad .$$

For an ideal phase-comparison system, this, of course, reduces to

$$(|f_1(a)| + |f_2(\alpha)|)^2 \quad .$$

The final estimate of ϕ is the value of $\hat{\phi}$ evaluated at the ML estimate of α.

In our second example there are no other parameters, but the sample pairs are modeled as incoherent, with as many unknown complex amplitudes as there are sample pairs. Specifically, the signal components are taken to be

$$E = z_k(n) = b(n)\, v_k(\gamma) \quad ,$$

with all the b(n) unknown, and the same noise model as in the previous example. Then

$$L^2 = L_o^2 + \sum_{n=1}^{n} |b(n)|^2 - 2 \text{ Re} \sum_{n=1}^{n} b(n)^* [v_1^*(\gamma) z_1(n) + v_2^*(\gamma) z_2(n)]$$

and the amplitude estimates are

$$b_1(n) = v_1^*(\gamma) z_1(n) + v_2^*(\gamma) z_2(n) \quad .$$

The estimator, $\hat{\gamma}$, will minimize

$$\text{Min } L^2 = L_o^2 - \sum_{n=1}^{n} |v_1^*(\gamma) z_1(n) + v_2^*(\gamma) z_2(n)|^2 \text{ , where the}$$

Min is taken over all values of the amplitude parameters.

We expand and write

$$\text{Min } L^2 = L_o^2 - |v_1(\gamma)|^2 \| z_1 \|^2 - |v_2(\gamma)|^2 \| z_2 \|^2 - 2 \text{ Re} \left\{ v_1^*(\gamma) v_2(\gamma)(z_2, z_1) \right\}$$

where

$$(z_2, z_1) \equiv \sum_{n=1}^{n} z_1(n) z_2^*(n) \quad .$$

The reduction of the data to an equivalent point on the sphere is less straightforward now, since

$$|(z_2, z_1)|^2 \neq \| z_1 \|^2 \| z_2 \|^2$$

except as a very special case. But substituting for $|v_1|^2$, $|v_2|^2$ and $v_1 v_2^*$ in terms of θ and ϕ , according to the formulas introduced in Section 2, we find

$$\text{Min } L^2 = L_o^2 - \frac{1}{2} \| z_1 \|^2 (1 - \cos\theta) - \frac{1}{2} \| z_2 \|^2 (1 + \cos\theta)$$

$$- \sin\theta \text{ Re} \left\{ e^{-i\phi}(z_2, z_1) \right\}$$

$$= \frac{1}{2} L_o^2 - \frac{1}{2} (\| z_2 \|^2 - \| z_1 \|^2) \cos\theta - \sin\theta |(z_2, z_1)| \cos(\phi - \psi) ,$$

where

$$\psi \equiv \arg (z_2, z_1) .$$

We define Q and η by the equations

$$Q \cos\eta \equiv \| z_2 \|^2 - \| z_1 \|^2$$

$$Q \sin\eta \equiv 2 |(z_2, z_1)| ,$$

and then

$$\text{Min } L^2 = \frac{1}{2} L_o^2 - \frac{1}{2} Q \cos\Delta ,$$

where

$$\cos\Delta = \cos\theta \cos\eta + \sin\theta \sin\eta \cos (\phi - \psi)$$

as before. Of course,

$$Q = \sqrt{(\| z_2 \|^2 - \| z_1 \|^2)^2 + 4 |(z_2, z_1)|^2} ,$$

and

$$\tan(\eta/2)\, e^{i\psi} = \frac{\sin \eta\, e^{i\psi}}{1 + \cos\eta} = \frac{2(z_2, z_1)}{Q + \|z_2\|^2 - \|z_1\|^2} .$$

When N=1, Q reduces to L_o^2 and the right side of the formula above becomes simply z_1/z_2, as it must.

For a general trajectory, the estimation of γ will be a very complicated function of the data, but for a great-circle or small-circle system, using phase-comparison coordinates, θ will be constant and $\hat{\phi}$ given by the relatively simple expression

$$\hat{\phi} = \psi = \arg (z_2, z_1) .$$

The corresponding detection statistic is

$$\frac{1}{2} L_o^2 + \frac{1}{2} Q \cos(\eta - \theta_o)$$

$$= \frac{1}{2} (|z_1|^2 + |z_2|^2) + \frac{1}{2} \cos \theta_o (|z_2|^2 - |z_1|)^2$$

$$+ \sin\theta_o |(z_2, z_1)|$$

$$= \sin^2(\theta_o/2) |z_1|^2 + \cos^2(\theta_o/2) |z_2|^2$$

$$+ 2 \sin(\theta_o/2) \cos(\theta_o/2) |(z_2, z_1)| \quad .$$

The ML estimator $\hat{\phi}$ can be expressed in terms of the individual estimators for each sample pair, namely

$$\hat{\phi}_n \equiv \arg \left\{ z_2^*(n) z_1(n) \right\} = \arg \left\{ \frac{z_1(n)}{z_2(n)} \right\} \quad ,$$

since clearly

$$\hat{\phi} = \arg \left\{ \sum_{n=1}^{n} |z_1(n) z_2(n)| \, e^{i\hat{\phi}_n} \right\} \quad .$$

This expression shows $\hat{\phi}$ as a nice mixture of coherent and incoherent processing. The individual ϕ - estimates are combined in an essentially coherent way, as a weighted sum, which is appropriate since the signal ratio is constant for all the pulses. The samples appear as weights in an incoherent form, since signal phase is unknown and variable from pulse to pulse.

6. GENERALIZATION OF THE NOISE MODEL

The remaining sections of Part I are concerned with non-white noise. The natural cause of correlated noise samples will be external sources of deliberate or accidental interference. Various noise models are discussed in this section, and it is shown that the general case is equivalent to a mixture of white internal noise and a single external noise source.

For the general noise covariance matrix,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} ,$$

the diagonal elements must be real and M_{21} must equal $M_{12}{}^{*}$, since M is Hermitian. Moreover, M is positive definite, which (in the two-dimensional case) is equivalent to the requirement of positive trace and positive determinant. If we write

$$M = \begin{bmatrix} a - b & ce^{i\phi} \\ ce^{-i\phi} & a+b \end{bmatrix} ,$$

with a, b, and c real, then we require

$$a > o$$

and

$$b^2 + c^2 < a^2 .$$

These conditions can be met by changing parameters again, introducing

$$b = d \cos\theta$$
$$c = d \sin\theta$$

and

$$a = d + f \quad ,$$

with d and f non-negative. Thus one general form for M is

$$M = \begin{bmatrix} f + d\,(1 - \cos\theta) & d \sin\theta\, e^{i\phi} \\ d \sin\theta\, e^{-i\phi} & f + d\,(1 + \cos\theta) \end{bmatrix}$$

with positive f and d and unrestricted values of θ and ϕ.

Now consider an external source of noise, at azimuth γ_J , so that in the absence of signal the total effective modulation at the antenna terminals is given by

$$Z_k = W_J(t)\, v_k(\gamma_J) + W_k(t) \quad ; \quad k = 1, 2 \; .$$

Here the $W_k(t)$ are white noise modulations as in Section 2, and $W_J(t)$ is an independent white noise process, with spectral density N_J :

$$E\, W_J(t) W_J^*(t') = 2\, N_J\, \delta(t-t').$$

Note that W_J is used with normalized beam gains.

Following Section 2, we assume a matched filter system, with output samples

$$z_k = j_k + n_k \quad .$$

The white noise components are unchanged, with

$$E|n_k|^2 = 2 \quad ,$$

and the interference components are

$$j_k = G\, v_k(\gamma_J) \int S^*(t) W_J(t) dt \quad .$$

It follows that these components are zero-mean, circular Gaussian samples with covariance matrix

$$E\, j_k\, j_\ell^* = v_k(\gamma_J) v_\ell^*(\gamma_J)\, G^2\, E\, \left| \int S^*(t)\, W_J(t) dt \right|^2$$

$$= \frac{2N_J}{N_o} \quad v_k(\gamma_J) v_\ell^*(\gamma_J) \quad ,$$

as a result of our filter gain normalization convention. The location of the interferor is characterized on the unit sphere by the point (θ_J, ϕ_J) , where

$$\frac{v_1(\gamma_J)}{v_2(\gamma_J)} \equiv \tan(\theta_J/2) e^{i\phi_J} \quad ,$$

so that

$$|v_1(\gamma_J)|^2 = \frac{1}{2} (1 - \cos \theta_J) \ ,$$

$$|v_2(\gamma_J)|^2 = \frac{1}{2} (1 + \cos \theta_J) \ , \quad \text{and}$$

$$v_1(\gamma_J) v_2^*(\gamma_J) = \frac{1}{2} \sin \theta_J e^{i\phi_J} \quad .$$

In terms of the parameter

$$J \equiv \frac{1}{2} \frac{N_J}{N_o} \ ,$$

the covariance matrix of the total noise is

$$M = 2 \begin{bmatrix} 1 + J\,(1 - \cos\theta_J) & J \sin \theta_J e^{i\phi_J} \\ \\ J \sin \theta_J\, e^{-i\phi_J} & 1 + J(1 + \cos\theta_J) \end{bmatrix} \quad .$$

Comparison with the general form shows that any noise model can be expressed as a superposition of white noise and an external source, provided that source can be placed anywhere on the unit sphere, and that a suitable scale factor, common to both channels, is introduced. This scale change will only affect the significance of the signal amplitude parameter.

It is not illogical to place the interferor off the characteristic trajectory of the system, since that trajectory may well be an idealization only valid in the system main lobe, and used only in the estimation of signal azimuth. In any case, as we shall see below, the resultant of a number of external sources, all on the trajectory, is itself off the trajectory except in a special case.

The significance of the parameter J, essentially a ratio of intensity of external to internal noise, may be seen by noting that

$$J = \frac{E\,(|j_1|^2 + |j_2|^2)}{E\,(|n_1|^2 + (n_2|^2)} \quad .$$

When signal components are added, they have the same form as before, namely

$$E\ z_k = b\ v_k(\gamma) \quad ,$$

where

$$|b|^2 = \frac{2\ E_s}{N_o} \quad ,$$

and E_s is still the total energy collected by the two beams. The effective signal-to-noise ratio of the sample pair, when both signal and interferor are present, may be taken to be

$$\mathrm{SNR} \equiv \frac{|Ez_1|^2 + |Ez_2|^2}{M_{11} + M_{22}} = \frac{|b|^2}{4(1 + J)}$$

$$= \frac{E_s}{2N_o + N_J}$$

By means of these relations, the theory can be applied to any model of signal and correlated noise, but we retain the notation of the noise covariance last given, in terms of J, θ_J and ϕ_J, because it is intuitively useful. Note that

$$\mathrm{Tr}\, M = 4(1 + J) \quad ,$$

and that

$$\mathrm{Det}(M) = 4(1 + 2J).$$

It is instructive to examine the case of multiple external sources, which are characterized by locations (θ_s, ϕ_s) and relative intensities J_s , where s runs from 1 to S. The sources are assumed to be independent, and the total covariance matrix, including normalized internal noise, will then have components

$$M_{11} = 2\left\{1 + \sum_{s=1}^{S} J_s(1 - \cos\theta_s)\right\} \quad ,$$

$$M_{22} = 2\left\{1 + \sum_{s=1}^{S} J_s(1 + \cos\theta_s)\right\} \quad ,$$

$$M_{12} = 2\sum_{s=1}^{S} J_s \sin\theta_s\, e^{i\phi_s} \quad , \text{ and}$$

$$M_{21} = M_{12}^{*} \; .$$

We equate this to the matrix

$$2k^2 \begin{bmatrix} 1 + J(1-\cos\theta_J) & J \sin\theta_J e^{i\phi J} \\ J \sin\theta_J e^{-\phi J} & 1 + J(1+\cos\theta_J) \end{bmatrix} ,$$

which corresponds to our standard form with a scale factor k on each channel. The multiple sources will thus correspond to an equivalent single source, together with a modified internal white noise component. The connection equations are

$$k^2 J \sin\theta_J e^{i\phi J} = \sum_{s=1}^{S} J_s \sin\theta_s s e^{i\phi s} ,$$

$$k^2 J \cos\theta_J = \sum_{s=1}^{S} J_s \cos\theta_s , \quad \text{and}$$

$$k^2 (1 + J) = 1 + \sum_{s=1}^{S} J_s .$$

Equating arguments on each side of the first equation yields ϕ_J, and the equality of the remaining magnitudes, together with the second equation, fixes θ_J and the product $k^2 J$. Finally, k^2 is found from the last equation.

It is interesting to find the location of the equivalent external source on the unit spere. Let $\underline{u}_J$ be a unit vector whose rectangular components are given by

$$(\underline{u}_J)_z = \cos\theta_J$$

$$(\underline{u}_J)_x = \sin\theta_J \cos\phi_J$$

$$(\underline{u}_J)_y = \sin\theta_J \sin\theta_J ,$$

and let $\underline{u}_s$, s=1, ...S, be unit vectors corresponding to the angles (θ_s,ϕ_s).

We equate real and imaginary parts of the two sides of our first equation above, and combine these two with the next equation, supplying appropriate unit vectors, to find

$$k^2 J \underline{u}_J = \sum_{s=1}^{S} J_s \underline{u}_s \quad .$$

Thus, for two sources, the resultant lies on the great circle arc joining the respective points on the surface of the sphere. This resultant is combined with a third source, and so on. The resultant of all sources lies within the smallest convex region which includes the individual locations.

With a group of external sources, each located on the system trajectory, the resultant will generally be off that trajectory. The one exception is the great circle characteristic.

7. MAXIMUM LIKELIHOOD ESTIMATION OF ANGLE IN THE PRESENCE OF INTERFERENCE

We have shown in Section 6 that the general case of correlated noise in the two monopulse channels is equivalent to a model having white noise components, together with a single external source of appropriate strength and location. Specifically, the complex samples upon which estimation will be based are taken to be

$$z_k = s_k + j_k + n_k \; ; \quad k = 1, 2 \quad ,$$

where

$$s_k = b \, v_k(\gamma) \quad ,$$

and

$$j_k = j_o v_k(\gamma_J) \quad .$$

Here b is an unknown complex signal amplitude, γ represents the signal angle (azimuth or elevation) to be estimated, γ_J is the known angle of the interference, and j_o is a complex, circular Gaussian variable representing the interference amplitude after filtering. The strength of this interference has been characterized by the parameter J, in such a way that

$$E|j_o|^2 = \frac{2N_J}{N_o} = 4\,J \quad .$$

As usual, the white noise terms n_1 and n_2 are independent, complex circular Gaussian variables, with

$$E|n_1|^2 = E|n_2|^2 = 2.$$

The particular definition used for J is influenced by the desire to keep the formulas which appear later in this section as simple as possible. Its relation to signal-to-noise ratio is discussed in Section 6. In this section, b is an unknown nuisance parameter, but its connection with signal energy and internal noise level given by

$$|b|^2 = \frac{2E_s}{N_o} \quad ,$$

is important in the analysis of estimator performance. With this model, the total noise covariance matrix is

$$M = 2 \begin{bmatrix} 1 + J(1 - \cos\theta_J) & J \sin\theta_J e^{i\phi_J} \\ J \sin\theta_J e^{-i\phi_J} & 1 + J(1 + \cos\theta_J) \end{bmatrix} \quad ,$$

where

$$\tan(\theta_J/2)e^{i\phi_J} = \frac{v_1(\gamma_J)}{v_2(\gamma_J)} \quad .$$

In order to solve the problem of estimating signal location in the presence of such correlated noise, assumed to be known, we seek a transformation, W, with the property that in the transformed coordinates,

$$z' = Wz \quad ,$$

the noise is white. The new covariance matrix will be

$$M' = Ez'(z')^{\dagger} = W M W^{\dagger} \quad ,$$

and we require that

$$M' = 2\ I_2 \quad ,$$

so that estimation problem in the new coordinates will be of exactly the type we have already discussed. The transformation matrix, W, cannot be unitary, since a unitary transformation would leave n statistically unchanged while making j look like a sample vector for an external source at another location. In fact, the transformed covariance matrix would look just like M, with the same value of J and a new location (θ'_J , ϕ'_J), fixed by the effect of the corresponding rotation on (θ_J , ϕ_J).

Since W is not unitary, we do not know, in general, what the characteristic trajectory of the system will be in the new, whitened, coordinates. Moreover, there are many choices for W, indicating the possibility of many different characteristics in the new coordinate system. Fortunately, all the whitened coordinate systems are simply related and, in fact, one can be transformed into any other by means of a unitary transformation. Thus, the various "whitened" characteristics on the sphere differ by simple rotations. To see this, note that if W is a whitening transformation, then

$$2\ W^{-1}(W^{\dagger})^{-1} = M \quad ,$$

and hence

$$W^{\dagger}W = 2M^{-1} \quad .$$

If V is another whitening transformation, then

$$V^{\dagger}V = W^{\dagger}W \quad ,$$

from which it follows that

$$W V^{-1} = (W^{-1})^{\dagger} V^{\dagger} \quad ,$$

which implies that

$$(V W^{-1})^{-1} = (V W^{-1})^{\dagger} \quad .$$

Thus $V W^{-1}$ is unitary and since

$$z'' \equiv V z = (V W^{-1}) z' \quad ,$$

the result is proved. This fact allows us to choose W in such a way that the characteristic trajectory is changed in a controlled and desirable way. When the estimator is then transformed back to the original coordinates and expressed in terms of the original data, the result will be independent of the choice made for W, so long as W whitens the noise.

Perhaps the simplest example of non-white noise is described by a diagonal covariance matrix. The noise components are independent, but have unequal variance. In terms of our general covariance matrix, the equivalent external source is at one of the poles of the sphere. Such a source would be in a null of one of the antenna beams. The simplest whitening matrix for this case is also diagonal,

$$W = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} ,$$

with the scaling factors, a and b, chosen to adjust each noise component to the standard variance, namely 2.

The mapping of points (θ, ϕ) into new points (θ', ϕ') on the sphere is very simple in this case. The general transformation equation reduces to

$$\tan(\theta'/2)e^{i\phi'} = \frac{a}{b}\tan(\theta/2)e^{i\phi} \quad ,$$

for this situation. The longitudes of points are unchanged ($\phi' = \phi$), while the latitudes are distorted by the non-linear transformation of the polar angles:

$$\tan(\theta'/2) = \frac{a}{b}\tan(\theta/2) \quad .$$

If, for example, a > b, then all points are moved toward the south pole, and points with small values of θ move more than points farther from the north pole.

A meridian, such as the characteristic trajectory of an ideal amplitude-comparison system, is invariant as a curve, but individual points are transformed according to the above equation. A parallel of latitude is displaced to a new latitude, but the relationship of points before and after the transformation is particularly simple, since longitude is preserved. Actually, this transformation (indeed, any linear transformation of the coordinates) takes circles on the sphere into other circles, but we can arrange things so that these general properties are not required to solve the estimation problem.

To deal with the general covariance matrix, we proceed indirectly, asking first what transformations (beside rotations) leave the equator of the sphere invariant as a curve. We intend to deal only with the case of an ideal phase-comparison system in general noise, since any system with a circular characteristic can be reduced to this case by a rotation followed by a diagonal transformation, and the resulting noise covariance matrix can then be represented in our general form. The effect of a general transformation,

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

on an equatorial point, $(\pi/2\ ,\phi)$, is given by the equation

$$\tan(\theta'/2)e^{i\phi'} = \frac{a\ e^{i\phi} + b}{c\ e^{i\phi} + d} .$$

If we choose $c = b^*$ and $d = a^*$, the right side can be written

$$\frac{a\ e^{i\frac{\phi}{2}} + b\ e^{-i\frac{\phi}{2}}}{b^* e^{i\frac{\phi}{2}} + a^*\ e^{-i\frac{\phi}{2}}} ,$$

which obviously has magnitude unity. Thus

$$W = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$$

is an equator-preserving transformation, and it is not hard to show that the most general transformation with this property has the form of W, followed by a rotation about the z-axis of the sphere. This last rotation is of no help in whitening the noise, hence we see what can be done with the form for W given above. Since a matrix to whiten M satisfies $W^\dagger W = 2M^{-1}$, we find out what W can do by computing

$$W^\dagger W = \begin{bmatrix} |a|^2 + |b|^2 & 2\ a^* b \\ 2\ ab^* & |a|^2 + |b|^2 \end{bmatrix} .$$

$$W^{\dagger}W = 2M^{-1} = \frac{1}{1+2J}\begin{bmatrix} 1 + J\,(1+\cos\theta_J) & -J\sin\theta_J e^{i\phi_J} \\ -J\sin\theta_J e^{-i\theta_J} & 1 + J(1-\cos\theta_J) \end{bmatrix}.$$

Let us define

$$\frac{b}{a} \equiv -\tan(\bar{\theta}/2)e^{i\bar{\phi}}$$

and put

$$|a|^2 + |b|^2 = 1 \quad ,$$

since a constant factor can be absorbed in the quantities d_+ and d_- . Then

$$|a|^2 = \cos^2(\bar{\theta}/2),\ |b|^2 = \sin^2(\bar{\theta}/2) \quad ,$$

$$2\,a^* b = -\sin\bar{\theta}e^{i\bar{\phi}} \quad , \text{ and}$$

$$W^{\dagger}W = \frac{1}{2}\begin{bmatrix} d_+ + d_- + (d_+ - d_-)\cos\bar{\theta} & -(d_+ + d_-)\sin\bar{\theta}e^{i\bar{\theta}} \\ -(d_+ + d_-)\sin\bar{\theta}e^{-i\bar{\phi}} & d_+ + d_- - (d_+ - d_-)\cos\bar{\theta} \end{bmatrix}$$

Comparison with the desired form shows that we must choose

$$\bar{\phi} = \phi_J \quad ,$$

and (equating the traces),

$$\frac{1}{2}(d_+ + d_-) = \frac{1+J}{1+2J}$$

Finally, we must have

$$\sin\bar{\theta} = \frac{J\sin\theta_J}{1+J}$$

and

$$\frac{1}{2}(d_+ - d_-) = \frac{1}{1+2J} \cdot \frac{J\cos\theta_J}{\cos\bar{\theta}}$$

We find $\bar{\theta}$ from the equation for $\sin\bar{\theta}$, taking the solution in the first quadrant, and solve for the diagonal factors:

$$d_{\pm} = \frac{1+J}{1+2J}\left\{1 \pm \frac{J\cos\theta_J}{(1+J)\cos\bar{\theta}}\right\}.$$

Notice that

$$\cos^2\bar{\theta} = 1 - \left(\frac{J\sin\theta_J}{1+J}\right)^2 = \frac{1+2J+J^2\cos^2\theta_J}{(1+J)^2},$$

so that

$$\cos\bar{\theta} = \frac{\sqrt{1+2J+J^2\cos\theta_J}}{1+J}.$$

The diagonal elements are equal in this matrix, and the same will be true of its inverse, hence W can only whiten noise covariance matrices which correspond to external sources on the equator (to make $\cos\theta_J = 0$). For a perfect phase-comparison system this would be the case, but as discussed in Section 6, we wish to analyze the model having an ideal phase-comparison trajectory but arbitrary noise.

Although a matrix like W cannot whiten the general noise covariance, it turns out that such a W can diagonalize the general M. To put it another way, M can be whitened by a sequence of two transformations, the first of which preserves the equator, while the second is an appropriate diagonal matrix. To see this, we write the general such two-stage transformation matrix in the form

$$W = \begin{bmatrix} \sqrt{d_+} & 0 & a & b \\ \\ 0 & \sqrt{d_-} & b^* & a^* \end{bmatrix} ,$$

where d_+ and d_- are additional parameters, which will be real and positive (complex factors can be reduced to this case by z-axis rotations). Again we compute $W^\dagger W$, and find

$$W^\dagger W = \begin{bmatrix} d_+|a|^2 + d_- |b|^2 & (d_+ + d_-)a^*b \\ \\ (d_+ + d_-)\, ab^* & d_+ |b|^2 + d_-|a|^2 \end{bmatrix} .$$

Now we have enough generality to match any noise covariance matrix, and we require that

The positive root reflects our earlier choice, and the formula shows that

$$\left| \frac{J \cos \theta_J}{(1 + J) \cos \bar{\theta}} \right| < 1 \quad ,$$

so that d_+ and d_- are positive.

The whitening properties of W depend only on the relative phase of a and b, and we choose to make a real, completing the specification of parameters in W. Altogether, we have

$$W = \frac{1}{\sqrt{1 + 2J}} \begin{bmatrix} \sqrt{1 + J + \dfrac{J \cos \theta_J}{\cos \bar{\theta}}} & 0 \\ 0 & \sqrt{1 + J - \dfrac{J \cos\theta_J}{\cos\bar{\theta}}} \end{bmatrix} \begin{bmatrix} \cos(\bar{\theta}/2) & -\sin(\bar{\theta}/2)e^{i\phi_J} \\ -\sin(\bar{\theta}/2)e^{-i\phi_J} & \cos(\bar{\theta}/2) \end{bmatrix} .$$

The first stage of this transformation preserves the equator; the second moves it to the circle parallel to the equator, a distance θ' from the north pole, where

$$\tan(\theta'/2) = \left\{ \frac{(1 + J)\cos \bar{\theta} + J \cos \theta_J}{(1 + J)\cos \bar{\theta} - J \cos \theta_J} \right\}^{1/2} = \sqrt{\frac{d_+}{d_-}} \quad .$$

Then

$$\cos\theta' = \frac{1 - \tan^2(\theta'/2)}{1 + \tan^2(\theta'/2)} = \frac{d_- - d_+}{d_+ + d_-} ,$$

or
$$\cos\theta' = - \frac{J \cos\theta_J}{(1 + J)\cos\bar{\theta}} .$$

If the external source is in the upper hemisphere, the whitened characteristic lies in the lower, and vice versa. The connection between longitudes on the original equator and the new characteristic in whitened coordinates is

$$e^{i\phi'} = \frac{\cos(\bar{\theta}/2)e^{i\phi} - \sin(\bar{\theta}/2)e^{i\phi_J}}{-\sin(\bar{\theta}/2)e^{i(\phi - \phi_J)} + \cos(\bar{\theta}/2)} ,$$

which can be written

$$e^{i(\theta' - \phi_J)} = \frac{e^{i\frac{\phi - \phi_J}{2}} - \tan(\bar{\theta}/2)e^{-i\frac{\phi - \phi_J}{2}}}{e^{-i\frac{\phi - \phi_J}{2}} - \tan(\bar{\theta}/2)e^{i\frac{\phi - \phi_J}{2}}} .$$

This in turn means that

$$\phi' - \phi_J = 2 \arg\left\{ e^{i\frac{\phi - \phi_J}{2}} - \tan(\bar{\theta}/2)e^{-i\frac{\phi - \phi_J}{2}} \right\} ,$$

and therefore that

$$\tan\left(\frac{\phi' - \phi_J}{2}\right) = \frac{1 + \tan(\bar{\theta}/2)}{1 - \tan(\bar{\theta}/2)} \tan\left(\frac{\phi - \phi_J}{2}\right) .$$

The point $\phi = \phi_J$ on the equator, and indeed the entire meridian containing the source location, is invariant to the whitening transformation, thanks to our choice of real a. The factor connecting the tangents can be simplified by squaring it, with the result that

$$\frac{1 + \tan(\bar{\theta}/2)}{1 - \tan(\bar{\theta}/2)} = \sqrt{\frac{1 + \sin\bar{\theta}}{1 - \sin\bar{\theta}}} .$$

In terms of the original noise parameters, the mapping of points $(\pi/2,\phi)$ on the original characteristic to their images, (θ',ϕ'), on the characteristic in whitened coordinates, is described by the equations

$$\cos\theta' = -\frac{J\cos\theta_J}{\sqrt{1 + 2J + J^2\cos^2\theta_J}} ,$$

$$\tan\left(\frac{\phi' - \phi_J}{2}\right) = \sqrt{\frac{1 + J(1 + \sin\theta_J)}{1 + J(1 - \sin\theta_J)}} \tan\left(\frac{\phi - \phi_J}{2}\right) .$$

A signal vector, bv, is changed by this transformation to b W v, which we write as

$$b' \, v' = b \, W \, v \quad .$$

The change in the amplitude parameter is needed since W is not unitary. The component ratio v_1/v_2 defines a point on the original trajectory:

$$\frac{v_1}{v_2} = e^{i\phi} \quad ,$$

while v_1'/v_2' defines the image point, as discussed above. The new vector, v', is normalized, hence the new signal amplitude satisfies

$$|b'|^2 = |b|^2 \left\{ |W_{11}v_1 + W_{12}v_2|^2 + |W_{21}v_1 + W_{22}\, v_2|^2 \right\} \quad .$$

We substitute for the components of W and note that

$$|v_1|^2 = |v_2|^2 = 1/2 \quad , \quad 2\, v_1\, v_2^* = e^{i\phi} \quad ,$$

with the results

$$|W_{11}v_1 + W_{12}v_2|^2 = \frac{1}{1+2J} \left(1 + J + \frac{J\cos\theta_J}{\cos\bar{\theta}}\right)\left[\frac{1}{2} - \frac{1}{2}\sin\bar{\theta}\,\cos(\phi - \phi_J)\right]$$

and

$$|W_{21}\, v_1 + W_{22}\, v_2|^2 = \left(\frac{1}{1+2J} \quad 1 + J - \frac{J\cos\theta_J}{\cos\theta}\right)\left[\frac{1}{2} - \frac{1}{2}\sin\bar{\theta}\,\cos(\phi - \phi_J)\right] .$$

Altogether, we find

$$b' = b \left\{ \frac{1}{1 + 2J} \left[1 + J - J \sin \theta_J \cos (\phi - \phi_J) \right] \right\}^{1/2} \quad ,$$

which shows that the new signal amplitude depends on the original signal location. This adds no complication to the estimation problem, since b' is just a nuisance parameter, but the relation of b' to b is required to evaluate the performance of the ML estimator in this case, which is done in Part II.

Now that we have a suitable whitening transformation, we can complete the analysis by finding the ML estimator of signal location in the new coordinates and transforming it back to the original set. For a single pulse, the data vector, z, is transformed to z' = Wz, and the ML estimator on the whitened sphere is

$$\hat{\phi}' = \arg (z'_1/z'_2) = \left\{\arg (z'_2)^* z'_1\right\} \quad ,$$

since the characteristic is a parallel of latitude. In terms of the original data components, we have

$$\arg \left\{(z'_2)^* z'_1\right\} = \arg \left\{(W_{21} z_1 + W_{22} z_2)^* (W_{11} z_1 + W_{12} z_2)\right\}$$

$$= \arg \left\{\left[- \sin(\bar{\theta}/2) e^{-i\phi_J} z_1 + \cos(\bar{\theta}/2) z_2\right]^* \left[\cos(\bar{\theta}/2) z_1 - \sin(\bar{\theta}/2) e^{i\phi_J} z_2\right]\right\} ,$$

since positive real factors do not affect the argument of the quantity on the right. Continuing, we get

$$\hat{\phi}' = \arg\Big\{ z_1 z_2^* \cos^2(\bar{\theta}/2) + z_1^* z_2 \sin^2(\bar{\theta}/2) e^{2i\phi_J}$$

$$- \sin(\bar{\theta}/2)\cos(\bar{\theta}/2)\ |z_1|^2 + |z_2|^2\ e^{i\phi_J} \Big\}$$

$$= \arg\Big\{ e^{i\phi_J} \Big[(1 + \cos\bar{\theta}) z_1 z_2^* e^{-i\phi_J} + (1 - \cos\bar{\theta}) z_1^* z_2\, e^{i\phi_J}$$

$$- \sin\bar{\theta}\ (|z_1|^2 + |z_2|^2) \Big] \Big\} \ .$$

Finally, we factor out the positive quantity, $\sin\bar{\theta}$, to obtain

$$\hat{\phi}' - \phi_J = \arg\Big\{ \cot(\bar{\theta}/2) z_1 z_2^* e^{-i\phi_J} + \tan(\bar{\theta}/2) z_1^* z_2\, e^{i\phi_J}$$

$$-(|z_1|^2 + |z_2|^2) \Big\} \ .$$

On the original sphere, the estimator, $\hat{\phi}$, is given by the inverse of the whitening transformation:

$$\tan\left(\frac{\hat{\phi} - \phi_J}{2}\right) = \sqrt{\frac{1 - \sin\bar{\theta}}{1 + \sin\bar{\theta}}}\quad \tan\left(\frac{\hat{\phi}' - \phi_J}{2}\right) \ .$$

The source longitude, ϕ_J, enters as a sort of reference angle in these expressions, which is a consequence of the fact that the source meridian is invariant to the whitening transformation. It is interesting that the other noise parameters, J and θ_J, enter these equations only through $\bar{\theta}$, so that the final estimator, $\hat{\phi}$, for the general case is identical in form to the estimator for a special case having the source on the equator at the same longitude and having an equivalent intensity, to maintain the value of $\bar{\theta}$. The performance of $\hat{\phi}$ will, of course, depend explicity on J and θ_J.

In the single-pulse case the formula for $\hat{\phi}'$ can be written so that it depends on the data only through the ratio z_1/z_2. However, in its present form, this expression is easily generalized to the case of multiple pulses with independent amplitudes, as in the second problem considered in Section 5. We assume the noise is stationary over the sequence of pulses, so the whitening transformation is the same for every pulse. Then $\hat{\phi}'$ is given by

$$\hat{\phi}' = \arg\left\{\sum_{n=1}^{n} z_2^{*'}(n)\, z_1'(n)\right\} \quad .$$

Repeating all the steps of the derivation just given, we easily find that

$$\hat{\phi}' - \phi_J = \arg\left\{\cot(\bar{\theta}/2)e^{-i\phi_J} \sum_{n=1}^{n} z_2^{*}(n)z_1(n) + \tan(\bar{\theta}/2)e^{i\phi_J} \sum_{n=1}^{n} z_1^{\dagger}(n)z_2(n) - \sum_{n=1}^{n} \left(|z_1(n)|^2 + |z_2(n)|^2\right)\right\}$$

In the notation of Section 5,

$$\sum_{n=1}^{n} z_2^{*}(n)z_1(n) = (z_2, z_1)$$

and

$$\sum_{n=1}^{n} |z_k(n)|^2 = \|z_k\|^2 \qquad ; \quad k = 1 ; 2 \quad ,$$

which allows us to write

$$\hat{\phi}' - \phi_J = \arg\left\{ \cot(\bar{\theta}/2)(z_2,z_1)\, e^{-i\phi_J} + \tan(\bar{\theta}/2)\,(z_2,z_1)^* e^{i\phi_J} - |z_1|^2 - |z_2|^2 \right\} .$$

The transformation from $\hat{\phi}$ to the desired $\hat{\phi}$ is, of course, the same as in the single-pulse problem.

These relations provide a formal solution to the multiple-pulse estimation problem in correlated noise. However, a more succinct result is possible by following a slightly different line of reasoning. Instead of locating the estimate as a point on the system trajectory in the whitened coordinates and then transforming it back to the original sphere (according to the equations just derived), it proves useful to consider the meridian on the whitened sphere which passes through the characteristic at the longitude, $\hat{\phi}$, of the estimate. This meridian maps back onto the original sphere as a circle, which must pass through the equator (the original characteristic) at the longitude $\hat{\phi}$. By pursuing the course, we obtain a simple, explicit formula for $\hat{\phi}$, and also a geometrical picture of the estimation process as a projection on the original sphere.

Without loss of generality we can take $\phi_J = 0$, since the original sphere can always be rotated to make this true, and the meridian corresponding to ϕ_J (and $\pi + \phi_J$) was preserved by the whitening transformation. Next, we introduce the notation

$$r \equiv \frac{2(z_2,\ z_1)}{|z_1|^2 + |z_2|^2} ,$$

and notice that the equation for $\hat{\phi}'$ can now be written

$$\hat{\phi}' = \arg\left\{\cot(\bar{\theta}/2)\, r + \tan(\bar{\theta}/2) r^* - 2\right\} .$$

Thus r is a sufficient statistic, in the original coordinates, for the estimation of signal location. The meridian through the ML estimator on the whitened sphere is the set of points

$$w' = \tan(\theta'/2)e^{i\hat{\phi}'}$$

for all values of θ'. Replacing $\tan(\theta'/2)$ by a real parameter k, and introducing the definition

$$\mu \equiv \tan(\bar{\theta}/2) \quad ,$$

we can write

$$w' = k \quad \frac{1}{\mu} r + \mu r^* - 2 \quad .$$

As k takes all positive and negative real values, w' traces out the desired meridian. The image of w' on the original sphere is w, determined by the inverse of the transformation W. It is easy to show that

$$\mathrm{Det}\ (W) = (1 + 2J)^{-1/2} \quad ,$$

and that, in the general case,

$$W^{-1} = \begin{bmatrix} \cos(\bar{\theta}/2) & \sin(\bar{\theta}/2)e^{i\phi_J} \\ \sin(\bar{\theta}/2)e^{-i\phi_J} & \cos(\bar{\theta}/2) \end{bmatrix} \begin{bmatrix} \sqrt{1 + J - \dfrac{J\cos\theta_J}{\cos\bar{\theta}}} & 0 \\ 0 & \sqrt{1 + J + \dfrac{J\cos\theta_J}{\cos\bar{\theta}}} \end{bmatrix}$$

We substitute in the inverse transformation

$$w = \frac{W_{11}^{-1}\, w' + W_{12}^{-1}}{W_{21}^{-1}\, w' + W_{22}^{-1}} \quad ,$$

putting $\phi_J = 0$ and using the notation d_+ and d_- introduced before. The result is

$$w = \frac{\sqrt{d_-}\,\cos(\bar{\theta}/2)w' + \sqrt{d_+}\,\sin\bar{\theta}/2)}{\sqrt{d_-}\,\sin(\bar{\theta}/2)w' + \sqrt{d_+}\,\cos(\bar{\theta}/2)} \quad .$$

Dividing numerator and denominator by $\sqrt{d_+}\cos(\theta/2)$ and recalling our definition for μ, we find

$$w = \frac{(d_-/d_+)^{1/2}\, w' + \mu}{(d_-/d_+)^{1/2}\, \mu\, w' + 1} \quad .$$

Now we substitute for w' , and absorb the quantity $(d_-/d_+)^{1/2}$ into the free parameter k (without changing notation), to find

$$w = \frac{k\left(\frac{1}{\mu}\; r + \mu r^* - 2\right) + \mu}{\mu k\left(\frac{1}{\mu}\; r + \mu r^* - 2\right) + 1} \quad .$$

This is a parametric representation of the curve on the original sphere which corresponds to the meridian of the estimator on the whitened sphere. For the moment, we put

$$\frac{1}{\mu} r + \mu r^* - 2 \equiv G \quad ,$$

so that

$$w = \frac{kG + \mu}{\mu kG + 1} \quad ,$$

and solve as follows:

$$- \frac{1}{k} = \frac{\mu w - 1}{w - \mu} G \quad .$$

The statement that the right side of this equation is real is the equation of our curve. It may be written

$$(w^* - \mu)(\mu w - 1) G = (w - \mu)(\mu w^* - 1) G^* \quad ,$$

or

$$\mu (G - G^*)(|w|^2 + 1) + (G^* - \mu^2 G) w - (G - \mu^2 G^*) w^2 = 0.$$

But

$$G - G^* = \left(\frac{1}{\mu} - \mu\right)(r - r^*) = 2i\left(\frac{1}{\mu} - \mu\right) \mathrm{Im}(r) \quad ,$$

and

$$G - \mu^2 G^* = \left(\frac{1}{\mu} - \mu^3\right) r - 2(1 - \mu^2)$$

$$= \mu \left(\frac{1}{\mu} - \mu\right) \left[\left(\frac{1}{\mu} + \mu\right) r - 2 \right] \quad ,$$

hence our equation simplifies to

$$2i\ \mathrm{Im}(r)(|w|^2 + 1) + \left[\left(\frac{1}{\mu} + \mu\right) r^* - 2\right] w$$

$$- \left[\left(\frac{1}{\mu} + \mu\right) r - 2\right] w^* = 0 .$$

Recalling the definition of μ, we find that

$$\frac{1}{\mu} + \mu = \cot(\bar{\theta}/2) + \tan(\bar{\theta}/2) = \frac{2}{\sin\bar{\theta}} ,$$

and therefore

$$|w|^2 + 1 + i\ \frac{(r - \sin\bar{\theta})\, w^* - (r^* - \sin\bar{\theta}) w}{\sin\bar{\theta}\ \ \mathrm{Im}(r)} = 0 .$$

It is easy to put this equation in the form

$$|w - w_o|^2 = R^2 ,$$

which represents a circle in the w - plane and also on the sphere, after substituting

$$w = \tan(\theta/2) e^{i\phi} .$$

However, we can better describe this circle by identifying certain points which must be on it. If w is real, our equation becomes

$$w^2 + 1 - \frac{2w}{\sin\bar{\theta}} = 0 ,$$

which is satisfied by the values

$$w_+ = \tan(\bar{\theta}/2) \text{ , and}$$

$$w_- = \cot(\bar{\theta}/2) \quad .$$

These are the values $w_+ = \mu$ and $w_- = 1/\mu$, which correspond to $w' = 0$ and $w' = \infty$, the poles of the whitened sphere. The image points, $w_\pm$, lie at equal distance from the poles of the original sphere, on the meridian $\phi = 0$. This is the meridian of the interferor, and w_+ represents the point $(\bar{\theta}, \phi_J)$ in general, while w_- represents $(\pi-\bar{\theta}, \phi_J)$. A line through w_+ and w_- is parallel to the north - south axis of the sphere, and planes through this line cut the sphere in all possible circles containing w_+ and w_- . Our equation describes one of these circles which passes through the equator at the point $\hat{\phi}$.

The circle also passes through a point which depends only on the data (note that w_+ and w_- depend only on the noise). To find this point, we must pick w proportional to r, since that will cause $\sin\bar{\theta}$ to drop out of the equation. Putting $w = kr$, with k real, we have

$$k^2|r|^2 + 1 - 2k = 0 \quad ,$$

or

$$k = 1 \pm \sqrt{1 - |r|^2} \quad ,$$

hence the points,

$$R_\pm = \frac{r}{1 \pm \sqrt{1 - |r|^2}}$$

lie on the circle. They are equidistant from the equator, at the same longitude. Thus $\hat{\phi}$ may be determined as the equatorial intercept of the circle through w_+ and w_- which passes through R_+ and R_-. If the interference goes away, i.e., $J \to 0$, then $\bar{\theta} \to 0$ and the points w_+ and w_- approach the poles. Circles through w_+ and w_- then become meridians, and $\hat{\phi}$ is simply $\arg(r) = \arg(z_2, z_1)$, the standard white noise solution. With interference present the lines of projection (the circles through w_+ and w_-) become distorted, moving $\hat{\phi}$ away from the value it would have in white noise.

To get an explicit expression for $\hat{\phi}$, we have only to put

$$w = e^{i\hat{\phi}} \; ,$$

to find where the circle intercepts the equator. Substitution yields

$$(r - \sin\bar{\theta})e^{-i\hat{\phi}} - (r^* - \sin\bar{\theta})e^{i\hat{\phi}} = 2i \sin\bar{\theta}\, \mathrm{Im}(r) \quad .$$

If

$$\alpha \equiv \arg(r - \sin\bar{\theta}) \quad ,$$

then

$$\sin(\alpha - \hat{\phi}) = \sin\bar{\theta} \; \frac{\mathrm{Im}(r)}{|r - \sin\bar{\theta}|} \qquad .$$

But $\mathrm{Im}(r) = \mathrm{Im}(r - \sin\bar{\theta})$, and hence

$$\sin(\alpha - \hat{\phi}) = \sin\bar{\theta} \;\; \sin\alpha \quad ,$$

or

$$\begin{aligned} \hat{\phi} &= \alpha - \sin^{-1}(\sin\bar{\theta} \sin\alpha) \\ &= \arg(r - \sin\bar{\theta}) - \sin^{-1} \left\{ \frac{\sin\bar{\theta}\ \mathrm{Im}(r)}{|r - \sin\bar{\theta}|} \right\} \quad . \end{aligned}$$

As noted before, $\hat{\phi} \to \arg(r)$ as $\sin\bar{\theta} \to 0$, which means that the $\sin^{-1}$ on the right side of the above equation must be taken in the range $(-\pi/2, \pi/2)$. In the general case, with interference at longitude ϕ_J, the result is

$$\sin(\alpha-\hat{\phi}) = \sin\bar{\theta}\,\sin(\alpha-\phi_J) \quad ,$$

with

$$\alpha \equiv \arg(r - \sin\bar{\theta} e^{i\phi_J}) \; .$$

In the single-pulse case, we have

$$r = \frac{2z_2^* z_1}{|z_1|^2 + |z_2|^2} = \frac{2(z/_1 z_2)}{1 + |z_1/z_2|^2} \quad ,$$

and it follows that

$$R_+ = z_1/\, z_2 = \tan(\eta/2)e^{i\psi} \quad ,$$

$$R_- = z_2^*/z_1^* = \cot(\eta/2)e^{i\psi} \; .$$

This simple result, that the projecting circle pass through the data point (η,ψ) itself in the one-pulse case, corresponds to the fact that, on the whitened sphere, the estimator lies on the meridian of the data points z_1'/z_2'.

It will be noted that our solution of the correlated-noise problem is based on what might be considered a fortunate guess for the whitening transformation. This approach was used because it minimizes the amount of tedious algebraic computation required. Another method, which involves a direct construction of the whitening transformation, depends upon the detailed correspondence between rotations of the sphere and unitary transformations, which is developed in Appendix A. Using this correspondence, one can write the whitening transformation as a sequence of three transformations, as follows. The first is a rotation which moves the location of the interferor to the north pole. The noise covariance is then diagonal and the characteristic is a great circle whose center is known. The second transform is a simple diagonal whitening operation which changes the characteristic to a small circle whose center and radius can be directly evaluated. The third transformation is another rotation, which moves the center of the characteristic circle to a pole, leaving us with standard white noise and a trajectory which is parallel of latitude. It is possible (but not simple) to show that the result is exactly the same as the transformation, W, used above.

Still another method is to begin with a triangular transformation matrix which removes from z_2 its projection on z_1, and equalizes the resulting noise variances. This operation leaves the noise white and the characteristic circular. By keeping track of what has happened to this circle, one can rotate it into position as a parallel of latitude. The result is again W, and in all cases the meridian of the interferor remains invariant.

8. NON-DETERMINISTIC SIGNAL MODELS

In all the cases so far considered, the signals have been modeled deterministically, but with unknown parameters according to various hypotheses. Instead, we can assume that the signal waveform is a random process, such as white noise, or simply postulate that the signal components of the sample pair (or pairs) are random variables with a corresponding moment matrix. We have already shwon (in Section 6) that the moment matrix of one sample pair takes the following form, when the only input is an external source of white noise:

$$M = 2 \begin{bmatrix} 1 + J(1 - \cos\theta_J) & J \sin\theta J \, e^{i\theta_J} \\ J \sin\theta J \, e^{-i\theta_J} & 1 + J(1 + \cos\theta_J) \end{bmatrix} .$$

The angles θ_J and ϕ_J locate the source on the unit sphere, and we now assume that this point lies on the characteristic trajectory of the system. This source location is to be estimated, and we allow the system trajectory to be completely general. The parameter J is the source-to-internal noise spectral density ratio:

$$J \equiv \frac{1}{2} \frac{N_J}{N_o} ,$$

and its significance was discussed in Section 6. The samples have been normalized so that $M = 2\, I_2$ in the absence of any external source, but the model for M is completely independent of the particular form of the filters used in the system. In another application, one might wish to estimate the azimuth of an external source with a signal waveform different from that for which the receiver channels are matched. This signal could be in the same band but utilize a different spreading code, for example. The response of the matched filters to a random code from this source could be modeled so that the samples are approximately Gaussian random variables, and a moment matrix of the form M would result, with an appropriate value of J.

Some of the oldest work on monopulse has used the random model, and we show here that the ML estimator of source location is identical to that obtained with deterministic models, regardless of the nature of the characteristic trajectory of the system.

With this random model, the pdf of the pair of samples, z_1 and z_2, is

$$f(z) = \frac{1}{\pi^2 \mathrm{Det}(M)} e^{-(z^\dagger M^{-1} z)} ,$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \text{ and } z^\dagger = \begin{bmatrix} z_1^*, z_2^* \end{bmatrix} ,$$

as before. Note that f(z) is the joint pdf of four real Gaussian variables. We already know that

$$\mathrm{Det}(M) = 4(1 + 2J) ,$$

and that

$$M^{-1} = \frac{1}{2(1 + 2J)} \begin{bmatrix} 1 + J(1 + \cos\theta_J) & - J \sin\theta_J \, e^{i\phi_J} \\ - J \sin\theta_J \, e^{i\phi_J} & 1 + J(1 - \cos\theta_J) \end{bmatrix} .$$

Substitution yields

$$\log f(z) = - \frac{1}{2(1 + 2J)} \Big\{ [1 + J(1 + \cos\theta_J)] |z_1|^2 + [1 + J(1 - \cos\theta_J)] |z_2|^2$$

$$- 2J \sin\theta_J \ \mathrm{Re} \left\{ z_1 z_2^* e^{-i\phi_J} \right\} \Big\} - \log(1 + 2J) - \log(4\pi^2).$$

With our standard notation for the data ratio:

$$\frac{z_1}{z_2} = \tan(\eta/2)e^{i\psi} \quad ,$$

we can use the expressions found in Section 2 for $|z_1|^2$, $|z_2|^2$ and $z_1 z_2^*$, with the result

$$\log f(z) = -\frac{1}{2}\frac{|z_1|^2 + |z_2|^2}{1 + 2J}\left[1 + J - J(\cos\theta_J\cos\eta + \sin\theta_J\sin\eta\cos(\phi_J - \psi))\right]$$

$$- \log(1 + 2J) - \log(4\pi^2).$$

Again, only the distance, Δ , between the points (θ_J , ϕ_J) and (η,ψ) on the unit sphere enters this expression, and

$$\log f(z) = -\frac{1}{2}\frac{|z_1|^2 + |z_2|^2}{1 + 2J}(1 + J - J\cos\Delta)$$

$$- \log(1 + 2J) - \log(4\pi^2) \quad .$$

The ML estimator of source location is the same as before, namely that point on the system trajectory nearest the data point, (η, ψ). If $\hat{\Delta}$ is the resulting minimum distance, then

$$\underset{\theta_J, \phi_J}{\text{Max}} \log f(z) = -\frac{1}{2}\frac{|z_1|^2 + |z_2|^2}{1 + 2J}\left[1 + 2J\sin^2\left(\frac{\hat{\Delta}}{2}\right)\right]$$

$$- \log(1 + 2J) - \log(4\pi^2) \ .$$

The estimate of J is then found by straightforward differentiation:

$$\hat{J} = \frac{1}{2}\left\{\frac{1}{2}(|z_1|^2 + |z_2|^2)\cos^2\left(\frac{\hat{\Delta}}{2}\right) - 1\right\} ,$$

or

$$\hat{N}_J = N_o\left\{\frac{1}{2}(|z_1|^2 + |z_2|^2)\cos^2\left(\frac{\hat{\Delta}}{2}\right) - 1\right\} .$$

For an ideal phase-comparison system, we have $\theta_J = \pi/2$, $\hat{\phi}_J = \psi$, and $\hat{\Delta} = \pi/2 - \eta$. Then

$$\cos^2\left(\frac{\hat{\Delta}}{2}\right) = \frac{1 + \cos\hat{\Delta}}{2} = \frac{1 + \sin\eta}{2}$$

and

$$\begin{aligned}
\hat{N}_J &= N_o\left\{\frac{1}{4}(|z_1|^2 + |z_2|^2)(1 + \sin\eta) - 1\right\} \\
&= N_o\left\{\frac{1}{4}(|z_1|^2 + |z_2|^2) + \frac{1}{2}|z_1 z_2^*| - 1\right\} \\
&= N_o\left\{\frac{1}{4}(|z_1| + |z_2|)^2 - 1\right\}
\end{aligned}$$

We recall that z_1 and z_2 were normalized filter outputs, using a convenient choice of filter gain. In terms of the quantities

$$H_k \equiv \int S^*(t) z_k(t) dt ,$$

which are the actual outputs of filters matched to S(t), we have

$$\hat{N}_J = \left(\frac{|H_1| + |H_2|}{2}\right)^2 \left\{\int |S(t)|^2 dt\right\}^{-1} - N_o .$$

This analysis is easily generalized to the case of N sample pairs, assumed independent of each other, and statistically identical to the single pair already modeled. If the n^{th} sample pair is denoted $z_1(n)$ and $z_2(n)$, as before, then the logarithm of the joint pdf of these samples is simply

$$\log f = -\frac{1}{2(1+2J)}\left\{[1 + J(1+\cos\theta_J)]\|z_1\|^2 + [1 + J(1-\cos\theta_J)]\|z_2\|^2 - 2J\sin\theta_J \,\mathrm{Re}\left\{e^{-i\phi_J}(z_2,z_1)\right\}\right\} - N\log(1+2J) - N\log(4\pi^2) \; .$$

In this expression we are using the vector space notation introduced in Section 5. As in that section, we define the effective data point, (η, ψ), by the equations

$$Q \sin\eta \, e^{i\psi} = 2\,(z_2,z_1)$$

$$Q \cos\eta = \|z_2\|^2 - \|z_1\|^2 \quad ,$$

with

$$Q = \left\{\left(\|z_2\|^2 - \|z_1\|^2\right)^2 + 4\left|(z_2,z_1)\right|^2\right\}^{1/2} \; .$$

In terms of these quanties, we can write

$$\log f = -\frac{1}{2(1+2J)}\left\{(1+J)\left[\|z_1\|^2 + \|z_2\|^2\right] - J\cos\Delta\right\} - N\log(1+2J) - N\log(4\pi^2) \; ,$$

where Δ is the distance from (θ_J, ϕ_J) to the point (η, ψ) on the unit sphere.

Again, the ML estimator of source location is the same as in the deterministic case discussed before, where every sample pair had an unknown amplitude and phase factor. If we define

$$P \equiv \|z_1\|^2 + \|z_2\|^2 \quad ,$$

we can express the interference-level estimator in the form

$$\hat{N}_J = \frac{N_o}{4N} (P + Q \cos\hat{\Delta}) - N_o \quad ,$$

where $\hat{\Delta}$ is the minimum distance to the system trajectory attained by the ML estimate of source location. For an ideal phase-comparison system, we again have $\cos\hat{\Delta} = \sin\eta$, and then

$$\hat{N}_J = N_o \left\{ \frac{1}{4N} \quad \|z_1\|^2 + \|z_2\|^2 + 2\left|(z_2,z_1)\right| \quad - 1 \right\} \quad ,$$

a direct generalization of the single-pair result.

Since we have modeled the external source waveform as random noise, we can interpret this source as an interferor instead of a desired signal, and make use of the above analysis to estimate its location. However, in this application we may wish to remove the constraint that the source lies on the system trajectory, since the source may be a composite of many separate interferors. In Section 6 we showed that the resultant may be off the characteristic in this case, and also that the multiple sources affect the level of the apparent white noise components.

With these considerations in mind we postulate that the noise covariance has the more general form

$$M = 2k^2 \begin{bmatrix} 1 + J(1 - \cos\theta_J) & J \sin\theta_J e^{i\phi_J} \\ J \sin\theta_J \, e^{-i\phi_J} & 1 + J(1 + \cos\theta_J) \end{bmatrix} \quad ,$$

where k is an additional parameter corresponding to an unknown white noise component. This is exactly the form used in Section 6 to represent the total noise equivalent to a collection of external sources and white internal noise. We assume that the data consists of N sample pairs and estimate (θ_J, ϕ_J) first, then J, and finally k. The intermediate results can be used if the subsequent parameters are presumed known. When all the parameters are unknown, M becomes an arbitrary positive-definite matrix, and our final result reproduces the known ML estimator of M, namely the sample covariance matrix.

The addition of the factor k^2 is a simple matter, and the logarithm of the joint pdf becomes

$$\log f = -\frac{1}{2k^2(1+2J)}\Big\{(1+J)[|z_1|^2 + |z_2|^2] + J\cos\theta_J[|z_1|^2 - |z_2|^2]$$

$$- 2J\sin\theta_J \,\mathrm{Re}\Big[e^{-i\phi_J}(z_2,z_1)\Big]\Big\} - N\log[k^4(1+2J)] - N\log(4\pi^2).$$

With the introduction of the data parameters (η, ψ), P and Q, as before, we obtain

$$\log f = -\frac{1}{2k^2(1+2J)}\Big\{(1+J)P - JQ\cos\Delta\Big\}$$

$$- N\log k^4(1+2J) - N\log(4\pi^2) \quad .$$

The source location, (θ_J, ϕ_J), is now unconstrained, hence we make the estimates

$$\hat{\theta}_J = \eta \ , \ \hat{\phi}_J = \psi \ ,$$

which corresponds to $\hat{\Delta} = 0$. With this choice, we have

$$\log f = -\frac{1}{2k^2(1+2J)}[P + J(P-Q)]$$

$$-N \log k^4(1+2J) - N\log(4\pi^2)$$

$$= -\frac{P+Q}{4k^2(1+2J)} - N\log(1+2J) - \frac{P-Q}{4k^2} - N\log k^4 - N\log(4\pi^2).$$

Differentiating with respect to $(1 + 2J)$, we find that the estimator of J satisfies the equation

$$1 + 2\hat{J} = \frac{P+Q}{4Nk^2} \quad ,$$

which leaves

$$\log f = -N - N\log\left(\frac{P+Q}{4N}\right) - \frac{P-Q}{4k^2} - N\log k^2 - N\log(4\pi^2).$$

If we stop at this point, i.e., treat k as known, the estimated covariance matrix is rather complicated, with elements

$$\hat{M}_{11} = \frac{P+Q}{4N} + k^2 + \frac{\|z_1\|^2 - \|z_2\|^2}{Q}\left(\frac{P+Q}{4N} - k^2\right)$$

$$\hat{M}_{22} = \frac{P+Q}{4N} + k^2 + \frac{\|z_2\|^2 - \|z_1\|^2}{Q}\left(\frac{P+Q}{4N} - k^2\right)$$

$$\hat{M}_{12} = \hat{M}_{21}^{*} = \frac{2}{Q}(z_2, z_1)\left(\frac{P+Q}{4N} - k^2\right) \quad .$$

If, however, we continue, k is estimated by maximizing the expression

$$- \frac{P - Q}{4\,k^2} - N \log k^2 \quad ,$$

with the result

$$\hat{k}^2 = \frac{P - Q}{4N} \quad .$$

When this estimate is substituted in $\hat{M}$, the complicated quantity Q disappears from the formulas and, substituting for P, we obtain the expected result

$$\hat{M} = \frac{1}{N} \begin{bmatrix} \|z_1\|^2 & (z_2,z_1) \\ & \\ (z_1,z_2) & \|z_2\|^2 \end{bmatrix} \quad .$$

APPENDIX A

THE CORRESPONDENCE BETWEEN UNITARY TRANSFORMATION IN THE SAMPLE SPACE AND ROTATIONS OF THE UNIT SPHERE

In Section 4 it was shown that a unitary matrix, V, viewed as a linear transformation in a two-dimensional complex vector space, corresponds to a rotation of the unit sphere. The mapping of a vector

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

to a point (θ,ϕ) on the sphere is expressed by the relation

$$\frac{z_1}{z_2} = \tan(\theta/2)\, e^{i\phi} \; .$$

Any linear transformation matrix can be multiplied by a complex scalar constant without affecting the transformation induced on the sphere, hence $e^{i\alpha}$ V and V correspond to the same rotation. In this case the scalar factor has magnitude unity to preserve the unitary character of the transformation. The correspondence of transformations can be made unique by requiring that Det(V) = 1, which we now assume.

To obtain the specific correspondence in question we consider, first, infinitesimal transformations. We write

$$V = I_2 + i\,\varepsilon\, H \;,$$

and find that

$$VV^{\dagger} = I_2 + \; i\,\varepsilon\,(H - H^{\dagger}) \;,$$

to first order in ε. Thus H must be Hermitian, and also, since

$$\mathrm{Det}\,(V) = 1 + i\,\varepsilon\,\mathrm{Tr}\,(H) \;,$$

to first order, the trace of H must be zero. Since ε is included as an explicit factor, the elements of H can be normalized in some way, and it proves convenient to use the following parametrization for V:

$$V = I_2 + (1/2)\varepsilon\, S(\underline{\mu}) \,,$$

where $$S(\underline{\mu}) \equiv \begin{bmatrix} \mu_3 & -(\mu_1 + i\mu_2) \\ -(\mu_1 - i\mu_2) & -\mu_3 \end{bmatrix}$$

and

$$\mu_1{}^2 + \mu_2{}^2 + \mu_3{}^2 \equiv 1 \,.$$

The numbers μ_1, μ_2 and μ_3 are real, and may be interpreted as the components of a unit vector $\underline{\mu}$ in a real, three-dimensional space. Obviously,

$$S^{\dagger} = S \quad ,$$

$$\mathrm{Tr}(S) = 0 \quad ,$$

and also, we find that

$$S^2 = I_2$$

The induced transformation on the sphere takes (θ,ϕ) into (θ',ϕ'), where

$$\tan(\theta'/2)e^{i\phi'} = \frac{V_{11}\tan(\theta/2)e^{i\phi} + V_{12}}{V_{21}\tan(\theta/2)e^{i\phi} + V_{22}} \,.$$

We write $\theta' = \theta + \Delta\theta$, $\phi' = \phi + \Delta\phi$ and expand the left side to first-order quantities. We also substitute for the components of V and develop the right side of the above equation to first order in ε. The result is

$$\tan(\theta/2) + \frac{1}{2}\sec^2(\theta/2)\Delta\theta + i\tan(\theta/2)\Delta\phi + \ldots$$

$$= \tan(\theta/2) + i\varepsilon\mu_3\tan(\theta/2) - (i\varepsilon/2)(\mu_1 + i\mu_2)e^{-i\phi}$$

$$+ (i\varepsilon/2)(\mu_1 - i\mu_2)\tan^2(\theta/2)e^{i\phi} + \ldots$$

or $$\Delta\theta = \varepsilon\,(\mu_2\cos\phi - \mu_1\sin\phi)$$

$$\Delta\phi = \varepsilon\,[\mu_3 - (\mu_1\cos\phi + \mu_2\sin\phi)\cot\theta] \quad .$$

Now if we write

$$\mu_3 = \cos\theta_o \quad , \quad \mu_1 + i\mu_2 = \sin\theta_o\, e^{i\phi_o} \quad ,$$

identifying the coordinate axes of the three-space in which $\underline{\mu}$ is defined with those of the space in which the unit sphere is embedded, we have

$$\Delta\theta = \varepsilon\,\sin\theta_o\,\sin(\phi_o - \phi)\ , \text{ and}$$

$$\sin\theta\Delta\phi = \varepsilon\,[\cos\theta_o\,\sin\theta - \sin\theta_o\,\cos\theta\,\cos(\phi_o - \phi)] \quad .$$

These relations show that the point

$$\theta = \theta_o \quad , \quad \phi = \phi_o$$

is unchanged by the transformation, and that its antipode

$$\theta = \pi - \theta_o\ , \quad \phi = \pi + \phi_o$$

is also invariant. Since V corresponds to a rotation, these points lie on its axis, which is to say that $\underline{\mu}$ itself defines the axis of rotation of the sphere. A plane, normal to $\underline{\mu}$ and passing through the center of the sphere defines a great circle, which is left invariant by the rotation. At the longitude $\phi = \phi_o + \pi$, this circle reaches its highest latitude, being tangent there to the circle $\theta = \frac{\pi}{2} - \theta_o$. At this vertex, we find

$$\Delta\theta = 0 \ , \quad \text{and}$$

$$\sin\theta \ \Delta\phi = \varepsilon \sin(\theta + \theta_o) \ , \quad \text{or}$$

$$\cos\theta_o \ \Delta\phi = \varepsilon \ ,$$

which shows that the angle of rotation corresponding to V is ε itself.

Now let $V(\underline{\mu}, \chi)$ be the unitary matrix corresponding to a rotation through a finite angle, χ, in the positive sense, about the axis $\underline{\mu}$. Then

$$V(\underline{\mu}, d\chi) = I_2 + (1/2)d\chi \ S(\underline{\mu}) \ .$$

The sequence of transformations described by the product $V(\underline{\mu}, d\chi) \ V(\underline{\mu},\chi)$ then corresponds to a rotation through χ about $\underline{\mu}$, followed by a rotation through $d\chi$ about the same axis. The resulting rotation, through $\chi + d\chi$ about $\underline{\mu}$ corresponds to $V(\underline{\mu}, \chi + d\chi)$, hence

$$V(\underline{\mu}, \chi + d\chi) = V(\underline{\mu}, d\chi) \ V(\underline{\mu},\chi)$$

$$= V(\underline{\mu},\chi) + (1/2)d\chi \ S(\underline{\mu}) \ V(\underline{\mu},\chi) \ .$$

Thus $V(\underline{\mu},\chi)$ satisfies the differential equation

$$\frac{d}{d\chi} V(\underline{\mu},\chi) = (1/2) \ S(\underline{\mu}) \ V(\underline{\mu},\chi)$$

with $V(\underline{\mu}, 0) = I_2$. The solution is simply

$$V(\underline{\mu}, \chi) = e^{(i/2)\chi S(\underline{\mu})} \cdot I_2$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!} \left(\frac{i\chi}{2}\right)^n [S(\mu)]^n \quad .$$

But $S^{2n} = I_2$ and $S^{2n+1} = S$, hence

$$V(\underline{\mu}, \chi) = \cos(\chi/2)\, I_2 + i \sin(\chi/2)\, S(\mu) \quad ,$$

or, in matrix form

$$V(\underline{\mu}, \chi) = \begin{bmatrix} \cos(\chi/2) + i\mu_3 \sin(\chi/2) & (-i\mu_1 + \mu_2) \sin(\chi/2) \\ (-i\mu_1 - \mu_2) \sin(\chi/2) & \cos(\chi/2) - i\mu_3 \sin(\chi/2) \end{bmatrix}$$

We can now use this correspondence to derive the transformation of the data vector which will convert an ideal amplitude-comparison system into an ideal phase-comparison one in the new coordinates. First, the sphere must be rotated through 90° about the X axis, in order to carry the amplitude-comparison trajectory (which lies in the XZ plane) into the equatorial plane. In the process, the north pole, which corresponded to the data point $(\Delta/\Sigma)=0$, is moved to the intersection of the negative Y axis with the sphere. A second rotation through 90°, this time about the new Z axis, will preserve the trajectory while moving this reference point to the nominal reference, $\theta=\pi/2$, $\phi=o$, for an ideal phase-comparison system. From the general form, a rotation about the X axis corresponds to

$$V_1(\chi) = \begin{bmatrix} \cos(\chi/2) & -i \sin(\chi/2) \\ -i \sin(\chi/2) & \cos(\chi/2) \end{bmatrix} \quad ,$$

and a Z-axis rotation is given by

$$V_3(\chi) = \begin{bmatrix} e^{i\chi/2)} & 0 \\ 0 & e^{-i\chi/2} \end{bmatrix} .$$

The complete transformation we require is $V_3(\frac{\pi}{2})\ V_1(\frac{\pi}{2})$, or

$$V = \begin{bmatrix} e^{i\pi/4} & 0 \\ & \\ 0 & e^{-i\pi/4} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ & \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= e^{-i\pi/4} \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} .$$

The new coordinates are then

$$Z'_1 = e^{-i\pi/4} \frac{1}{\sqrt{2}} (Z_2 + iZ_1)$$

$$Z'_2 = e^{-i\pi/4} \frac{1}{\sqrt{2}} (Z_2 - iZ_1) .$$

The phase factor, $e^{-i\pi/4}$, is completely harmless and can be dropped, which completes the derivation since Z_1 was the difference channel, Z_2 the sum channel sample in our amplitude-comparison formulation.

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Maximum likelihood methods are applied to a series of monopulse problems, involving both angle estimation and signal detection. Only the two-beam, off-boresight monopulse problem is studied. Explicit maximum likelihood estimators are obtained in Part I, and their probability distributions will be discussed in the forthcoming Part II. Both deterministic and stochastic signal models are used here, and maximum likelihood estimates are obtained for the single pulse case and for different models of the multiple pulse problem. Particular emphasis is given to the problem of angle estimation in correlated noise, representing the case of arbitrary interference.

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